THE USE OF BILINEAR MAPPINGS IN THE CLASSIFICATION OF GROUPS OF CLASS 2

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Introduction. Division of the set of all groups into families provides a convenient method of classifying groups [1].[1] We show that there is a natural one-one correspondence between the set of all families of groups of class 2, plus the family of all Abelian groups, and the set of all families of "regular bilinear mappings."

In the second section, products of families are defined and it is shown that the family of any finite group of class 2 may be expressed uniquely as a product of indecomposable mappings.

1. Regular bilinear mappings and groups of class 2. In this paper, H and K will always be Abelian groups. We say that f is a regular bilinear mapping of H into K, written

(1) $$f \in \mathfrak{M}(H, K),$$

if, to every ordered pair (x, y), x and y in H, there is defined a unique $f(x, y)$ in K and, for all x, x', y, y' in H,

(2) $$f(xx', y) = f(x, y)f(x', y),$$

(3) $$f(x, yy') = f(x, y)f(x, y'),$$

(4) $$f(x, x) = e \qquad (e \text{ is always the group identity}),$$

(5) if, for all y, $f(x, y) = e$, then $x = e$,

(6) $$K = \{f(x, y)\}.$$

It is easy to show that these conditions imply that H and K are Abelian.

G, Z, and Q will always stand for a group, its centre, and its commutator subgroup respectively and we shall further assume that G is either Abelian or of class 2, that is

(7) $$Q \subset Z.$$

If we define f by

(8) $$f(Zx, Zy) = [x, y] = x^{-1}y^{-1}xy,$$

it is easy to show that $f \in \mathfrak{M}(G/Z, Q)$. We write $f = M(G)$.

We say that f and f' in $\mathfrak{M}(H, K)$ and $\mathfrak{M}(H', K')$, respectively, be-

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[1] Number **1** in brackets refers to the reference at the end of the paper.

long to the same family, denoted by $f \sim f'$, if and only if there exist isomorphisms ϕ and ψ of H into H' and K into K' respectively and such that, for all x and y in H,

$$f'(\phi x, \phi y) = \psi f(x, y). \tag{9}$$

The condition [1] that G and G' belong to the same family, denoted by $G \sim G'$, is precisely that

$$f = M(G) \sim f' = M(G'). \tag{10}$$

If $\mathfrak{F} = F(f)$ and $\mathfrak{G} = F(G)$ denote the families to which f and G respectively belong and we define $M(\mathfrak{G})$ by

$$M(\mathfrak{G}) = F(M(G)), \qquad \text{for any } G \text{ in } \mathfrak{G}, \tag{11}$$

we then have the following theorem.

THEOREM 1. *The correspondence $\mathfrak{G} \rightarrow M(\mathfrak{G})$ is one-one between the set of all families of groups of class 2, plus the family of all Abelian groups, and the set of all families of regular bilinear mappings.*

This follows at once from the above definitions and Theorem 2.

The following proof of Theorem 2 is due to Professor Saunders MacLane and replaces a much longer proof[2] based on construction using a special basis.

THEOREM 2. *For any f in $\mathfrak{M}(H, K)$, there exists G such that $f \sim M(G)$ and $Z = Q$.*

The group G required consists of all ordered pairs (x, h) with x in H and h in K and multiplication is defined by

$$(x, h)(y, k) = (xy, hkg(x, y)), \tag{12}$$

where $g(x, y)$ satisfies the following conditions:

$$g(x, y) = f(x, y)g(y, x), \tag{13}$$

$$g(x, y)g(xy, z) = g(y, z)g(x, yz), \tag{14}$$

$$g(e, e) = e. \tag{15}$$

A factor set g may be found by transfinite induction by the following lemma.

LEMMA 1. *If $R < S \leqq H$ and S/R is cyclic and $h(x, y)$ is defined in R and satisfies* (13), (14), *and* (15), *then we can find $g(x, y)$ defined in S, satisfying* (13), (14), *and*

[2] Given in an earlier version of this paper.

(16) $$g(x, y) = h(x, y), \qquad \textit{for all } x, y \textit{ in } R.$$

Let Rz generate S/R; if S/R has finite order n, $z^n = a \in R$, we define $g(x, y)$, $x = x'z^i$, $y = y'z^j$, x', y' in R, $0 \leqq i, j < n$, by

(17) $$g(x, y) = h(x', y')f(z^i, y'), \qquad \text{if } i + j < n,$$

and by

(18) $$g(x, y) = h(x', y')f(z^i, y')h(x'y', a), \qquad \text{if } i + j \geqq n.$$

If S/R has infinite order, we define $g(x, y)$ by (17) for all i, j.

It is not difficult to verify that $g(x, y)$ has the required properties, (14) has six cases to be considered, when n is finite.

2. **Direct products of multi-linear mappings and groups.** We say that f is a regular multi-linear mapping on $H, K, \cdots,$ to L, denoted by $f \in \mathfrak{M}(H, K, \cdots; L)$, where $H, K, \cdots$ is a set of n groups $(2 \leqq n < \infty)$, if, to every x in H, y in $K, \cdots,$ there is defined $f(x, y, \cdots)$ in L such that

(1) $f(xx', y, \cdots) = f(x, y, \cdots)f(x', y, \cdots)$, and so forth,

(2) if $f(x, y, \cdots) = e$ for all $y, \cdots$, then $x = e$, and so forth,

(3) $$L = \{f(x, y, \cdots)\}.$$

We shall always used x, y to stand for elements of H, K and extend the convention to groups distinguished by suffixes.

It easily follows that $H, K, \cdots$ are Abelian and, for $n = 2$, L also.

We say that f is the direct product of f_i $(i \in I)$ or

(4) $$f = \prod_{i \in I} f_i, \qquad f_i \in \mathfrak{M}(H_i, K_i, \cdots; L_i),$$

if

(5) $$H = \prod_{i \in I} H_i, \ K = \prod_{i \in I} K_i, \cdots, L = \prod_{i \in I} L_i \qquad \text{(direct products)}$$

and

(6) $f(x_i, y_p, \cdots) = f_i(x_i, y_i, \cdots)$, if $i = p = \cdots$, and e otherwise.

It is easy to obtain the following result.

LEMMA 2. *If* $f = M(G)$, $f_i = M(G_i)$ $(i \in I)$, *then* $G \sim \prod_{i \in I} G_i$ *if and only if* $f \sim \prod_{i \in I} f_i$.

We now come to the main result on products of mappings.

THEOREM 3. *If* $f \in \mathfrak{M}(H, K, \cdots; L)$, $f = \prod_{i \in I} f_i$, $f_i \in \mathfrak{M}(H_i, K_i, \cdots; L_i)$, *and also* $f = \prod_{j \in J} f_j'$, $f_j' \in \mathfrak{M}(H_j', K_j', \cdots; L_j')$ *then*

$f=\prod_{i\in I,j\in J} f_{ij}$, $f_{ij}\in\mathfrak{M}(H''_{ij}, K''_{ij}, \cdots; L''_{ij})$, $H''_{ij}=H_i\cap H'_j$, $\cdots$, $L''_{ij}=L_i\cap L'_j$.

By (5), $x=\prod_{i\in I} x_i$, $x_i=\eta_i x$, where η_i is a homomorphism on H to H_i. We have similarly

$$y = \prod_{i\in I} \eta_i y, \qquad \text{and so forth,}$$

$$x = \prod_{j\in J} x'_j, \qquad x'_j = \theta_j x, \qquad \text{and so forth.}$$

We now have

$$x = \prod_{i\in I, j\in J} x_{ij}, \quad x_{ij} = \theta_j \eta_i x \in H''_{ij} = \theta_j H_i. \tag{7}$$

LEMMA 3. *$f(x_{ij}, y_{pq}, \cdots)=e$ unless $i=p=\cdots$ and $j=q=\cdots$.*

By (6), applied to the f' product, the second equation follows at once. If the first equation is false, let $x_{ij}=\theta_j x_i$, and so forth. Then from (1), (5), and (6)

$$f(x_i, y_p, \cdots) = \prod_{k\in J} f(\theta_k x_i, \theta_k y_p, \cdots) = e,$$

so that $f(\theta_k x_i, \theta_k y_p, \cdots)=e$, for all k in J, in particular j, so that $f(x_{ij}, y_{pj}, \cdots)=e$ as required.

We now show that the expression (7) of x as a product of x_{ij} is unique.

Suppose then that $e=\prod_{i\in I,j\in J} x_{ij}$, $x_{ij}\in H''_{ij}$. We then have

$$f(x_{ij}, y_{pq}, \cdots) = e \tag{8}$$

for all $i, p, \cdots$ in I, $j, q, \cdots$ in J, all y_{pq} in K_{pq}, and so forth, since (8) holds at once, by Lemma 3, unless both $i=p=\cdots$ and $j=q=\cdots$, while in this case, by (1),

$$e = f(e, y_{pq}, \cdots) = \prod_{l\in I, m\in J} f(x_{lm}, y_{pq}, \cdots) = f(x_{ij}, y_{pq}, \cdots),$$

as, by Lemma 3, $f(x_{lm}, y_{pq}, \cdots)=e$, unless $l=i$ and $m=j$.

Then, by (1), (7), and (8),

$$f(x_{ij}, y, \cdots) = e, \qquad \text{for all } y \text{ in } K \text{ and so forth,} \tag{9}$$

so that by (2), we have $x_{ij}=e$, and H is the direct product

$$H = \prod_{i\in I, j\in J} H''_{ij}. \tag{10}$$

We now show that $H''_{ij} = H_i \cap H'_j$. First,

(11) $$x_{ij} = \theta_j x_i \in H'_j.$$

Let

(12) $$x_{ij} = \prod_{p \in I} x'_p, \qquad x'_p \in H_p,$$

then

(13) $$x_{ij} = \theta_j x_{ij} = \prod_{p \in I} \theta_j x'_p, \qquad \theta_j x'_p \in H''_{pj},$$

(14) $$e = \theta_k x_{ij} = \prod_{p \in I} \theta_k x'_p, \qquad \theta_k x'_p \in H''_{pk}, \qquad k \neq j.$$

By (10), we have

(15) $$\theta_q x'_p = x_{ij}, \text{ if } p = i \text{ and } q = j, \text{ and } e \text{ otherwise,}$$

so that,

(16) $$x_{ij} = x'_i \in H_i,$$

as required. Conversely, if $x \in H_i \cap H'_j$,

(17) $$x = \theta_j \eta_i x \in H''_{ij}.$$

We thus have

(18) $$H''_{ij} = H_i \cap H'_j.$$

By Lemma 3, (1), (3), (10), (5), and (18) the proof of Theorem 3 is easily completed.

We can define products of families of n-linear mappings by

(19) $$\mathfrak{F} = \prod_{i \in I} \mathfrak{F}_i = F\left(\prod_{i \in I} f_i \right), \qquad \mathfrak{F}_i = F(f_i),$$

and, similarly, products of families of groups of class 2 or 1, by

$$\mathfrak{G} = \prod_{i \in I} \mathfrak{G}_i \text{ if } G \sim \prod_{i \in I} G_i, \text{ for } G \text{ in } \mathfrak{G} \text{ and } G_i \text{ in } \mathfrak{G}_i.$$

The products are clearly independent of the particular elements by which they are defined. A family is indecomposable if it cannot be expressed as a product of nontrivial factors. We easily obtain from the above results the following theorem.

THEOREM 4. *If* $\mathfrak{F} = \prod_{i \in I} \mathfrak{F}_i = \prod_{j \in J} \mathfrak{F}'_j$, *then* $\mathfrak{F} = \prod_{i \in I, j \in J} \mathfrak{F}''_{ij}$, *where* $\mathfrak{F}_i = \prod_{j \in J} \mathfrak{F}''_{ij}$ *and* $\mathfrak{F}'_j = \prod_{i \in I} \mathfrak{F}''_{ij}$.

Further $\mathfrak{F}$ can, apart from trivial (unit) factors, be expressed in at most one way as a product of indecomposable factors. The same results hold for families of regular bilinear mappings ($n=2$, $H=K$, $H_i=K_i$) and for families of groups of class 1 or 2. If either Q or G/Z be finite, $F(G)$ is uniquely expressible as a product of indecomposable families.

Reference

1. P. Hall, *Classification of prime power groups*, Journal für Mathematik (1940).

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A SHORT PROOF OF AN IDENTITY OF EULER

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Euler discovered the identity

$$\prod_{s=1}^{\infty}(1-x^s) = 1 + \sum_{s=1}^{\infty}(-1)^s\left[x^{s(3s-1)/2} + x^{s(3s+1)/2}\right]. \tag{1}$$

He used it in the theory of partitions, and, after some time, he proved it [1].[1] Later, famous proofs involving theta functions and combinatorial arguments were given by Jacobi and F. Franklin [2]. The following algebraic proof is quite simple.

Let the partial products and partial sums of (1) be

$$P_0 = 1, \qquad P_n = \prod_{s=1}^{n}(1-x^s),$$

and

$$S_n = 1 + \sum_{s=1}^{n}(-1)^s\left[x^{s(3s-1)/2} + x^{s(3s+1)/2}\right].$$

Then S_n and P_n are related by the *finite* identity

$$S_n = F_n \qquad \text{where } F_n = \sum_{s=0}^{n}(-1)^s\frac{P_n}{P_s}x^{sn+s(s+1)/2}. \tag{2}$$

To prove (2) we detach the last term, $s=n$, and split the remaining sum into two parts by putting $P_n = P_{n-1} - x^n P_{n-1}$. This gives

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[1] Numbers in brackets refer to the references cited at the end of the paper.