ELEMENTARY DIVISORS OF AB AND BA

HARLEY FLANDERS

This note contains a complete description of the relation between AB and BA for arbitrary matrices A, B, and certain related results.[1]

Let L, M be vector spaces over a field k. We shall denote the dimension of L over k by $d[L]$, the space of linear transformations (homomorphisms) of L into M by Hom (L, M), and also write End $(L)=$Hom (L, L) for the space of all linear tranformations (endomorphisms) on L into itself. If $A\in$Hom (L, M), then the effect of A on a vector $x\in L$ will be denoted by xA, the range of A by LA, the null space of A by $\mathfrak{N}(A)$, and the nullity of A by $\nu(A)=d[\mathfrak{N}(A)]$. All linear spaces considered here will be finite-dimensional over their scalar field k, which may be arbitrary.

If $A\in$End (L), $B\in$End (M), we shall say that A and B are *similar* if there is an isomorphism P on L onto M such that $A=PBP^{-1}$. This is a very mild generalization of the usual notion of similarity, and it is easy to see that the standard theorems of the similarity theory of matrices are valid with appropriate modifications in their statements. For example, the elementary divisors of A form a complete set of invariants of A under the equivalence relation of similarity.

If $A\in$End (L) and if an element α of k is a characteristic root of A, we shall understand by the *geometric multiplicity* of α the integer $d[\mathfrak{N}(\alpha I-A)]$ where I denotes the identity endomorphism. The *algebraic multiplicity* of α is the number of times α is a root of the characteristic function of A.

Now let $d[L]=m$, $d[M]=n$, $A\in$Hom (L, M), $B\in$Hom (M, L). Then $AB\in$End (L), $BA\in$End (M). It is known that the nonzero characteristic roots of AB and BA coincide with the same algebraic multiplicities.

A first approximation of our main result is given in the following theorem.

THEOREM 1. *The geometric multiplicities of the nonzero characteristic roots of AB coincide with those of BA.*

Let $\alpha\neq 0$. We must prove that $\nu(\alpha I-AB)=\nu(\alpha I-BA)$. If $x\in\mathfrak{N}(\alpha I-AB)$, then $(xA)(\alpha I-BA)=x(\alpha I-AB)A=0$. Thus

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[1] The author is indebted to Professor I. Kaplansky for suggesting this investigation.

$xA \in \mathfrak{N}(\alpha I - BA)$, $\mathfrak{N}(\alpha I - AB)A \subset \mathfrak{N}(\alpha I - BA)$, $\mathfrak{N}(\alpha I - AB)AB \subset \mathfrak{N}(\alpha I - BA)B \subset \mathfrak{N}(\alpha I - AB)$. This means that AB induces an endomorphism on $\mathfrak{N}(\alpha I - AB)$. This endomorphism is actually nonsingular, for if $x(\alpha I - AB) = 0$ and $xAB = 0$, then $\alpha x = 0$, $x = 0$. Thus $\mathfrak{N}(\alpha I - AB)AB = \mathfrak{N}(\alpha I - AB)$, $\mathfrak{N}(\alpha I - BA)B = \mathfrak{N}(\alpha I - AB)$, $\nu(\alpha I - AB) \leqq \nu(\alpha I - BA)$. By symmetry, $\nu(\alpha I - AB) = \nu(\alpha I - BA)$.

Now let $A \in \text{End}\ (L)$ and assume that 0 is a characteristic root of A. Let $t^{n_1}, t^{n_2}, \cdots$ be the sequence of elementary divisors of A corresponding to this root, and write down the sequence $n_1 \geqq n_2 \geqq \cdots$, made infinite by adjunction of zeros. Then A is completely described (up to similarity) by (a) the elementary divisors of A which do not have zero as a root and (b) the sequence $n_1, n_2, \cdots$. We have the following theorem.

THEOREM 2. *The elementary divisors of AB which do not have zero as a root coincide with those of BA.*

If $n_1 \geqq n_2 \geqq \cdots$ is the sequence constructed above for AB and $n_1' \geqq n_2' \geqq \cdots$ that for BA, then $|n_j - n_j'| \leqq 1$. Conversely, if $C \in \text{End}\ (L)$, $D \in \text{End}\ (M)$ and these conditions are satisfied, then there exist $A \in \text{Hom}\ (L, M)$ and $B \in \text{Hom}\ (M, L)$ such that $AB = C$ and $BA = D$.

Our proof is based on Fitting's Lemma.[2] Take $p \geqq m$, $p \geqq n$ and set $C = AB$, $D = BA$. Then $L = LC^p \oplus \mathfrak{N}(C^p)$, $M = MD^p \oplus \mathfrak{N}(D^p)$, C acts as an automorphism on LC^p and is nilpotent on $\mathfrak{N}(C^p)$. We easily see that A induces an isomorphism A' on LC^p onto MD^p. The relation $A(BA) = (AB)A$ implies that $A'D = CA'$. This means that the contraction of C to LC^p is similar to the contraction of D to MD^p. But the effect of C on LC^p completely determines (and is determined by) the elementary divisors of C which do not have zero as a root, by Fitting's Lemma.

We may now assume that C and D are nilpotent. Let $L_j = \mathfrak{N}(C^j)$, $M_j = \mathfrak{N}(D^j)$. It is not hard to see that our assertion on the n_i and n_i' is equivalent to the relations

$$d[L_{j+1}] - d[L_j] \leqq d[M_j] - d[M_{j-1}],$$
$$d[M_{j+1}] - d[M_j] \leqq d[L_j] - d[L_{j-1}]$$

for $j \geqq 1$, where we set $M_0 = L_0 = 0$. In fact, $d[L_j] - d[L_{j-1}]$ is the number of n_i which are not less than j.[3] To prove, say, the first of these, we note that $\mathfrak{N}(A) \subset \mathfrak{N}(C) \subset L_j$, hence $d[L_j] - \nu(A) = d[L_jA] = d[LA \cap \mathfrak{N}(D^{j-1}B)]$ by the homomorphism principle. We apply the

[2] N. Jacobson, *The theory of rings*.

[3] See, for example, van der Waerden, *Moderne Algebra*, vol. 2, 2d ed., p. 115.

homomorphism D^{i-1} to the space $LA \cap \mathfrak{N}(D^{i-1}B)$ to obtain

$$d[LA \cap \mathfrak{N}(D^{i-1}B)] = d[LA \cap M_{j-1}] + d[LAD^{i-1} \cap \mathfrak{N}(B)].$$

We next write down the same relations for index $j+1$, subtract, and, observing that $LAD^i = LCAD^{i-1} \subset LAD^{i-1}$, obtain

$$d[L_{j+1}] - d[L_j] \leqq d[LA \cap M_j] - d[LA \cap M_{j-1}] \leqq d[M_j] - d[M_{j-1}].^4$$

In order to prove the sufficiency of our conditions, we first observe that if A is replaced by PAQ^{-1} and B by QBP^{-1}, then AB is replaced by $P(AB)P^{-1}$ and BA by $Q(BA)Q^{-1}$. Thus it suffices to find A, B such that AB is similar to C and BA to D. By virtue of Fitting's Lemma, we may decompose A and B into direct sums $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$ so that $A_iB_i = C_i$, $B_iA_i = D_i$. We easily take care of the parts C_1, D_1 which are automorphisms and hence, by hypothesis, similar transformations. The case in which C and D are nilpotent remains. We use the classical canonical decomposition of L, M:

$$L = N_1 \oplus N_2 \oplus \cdots, \qquad M = N_1' \oplus N_2' \oplus \cdots.$$

We shall define A, B in such a manner that $N_jA \subset N_j'$, $N_j'B \subset N_j$. Thus we are reduced to the case in which L, M are indecomposable, L has a basis $x, xC, \cdots, xC^{m-1}$, M has a basis $y, yD, \cdots, yD^{n-1}$, $xC^m = yD^n = 0$, and $|m-n| \leqq 1$. If $m=n$, A and B are defined by

$$xA = y, \quad XCA = yD, \cdots, xC^{m-1}A = yD^{m-1},$$
$$yB = xC, \quad yDB = xC^2, \cdots, yD^{m-2}B = xC^{m-1}, \quad yD^{m-1}B = 0.$$

If $m > n$, hence $m = n+1$, A and B are defined by

$$xA = y, \quad xCA = yD, \cdots, xC^{n-1}A = yD^{n-1}, \quad xC^nA = 0,$$
$$yB = xC, \quad yDB = xC^2, \cdots, yD^{n-1}B = xC^n.$$

In either case we easily verify that $AB = C$ and $BA = D$.

The following result[5] is due to W. T. Reid. Let A be $m \times n$; B, $n \times m$; N, $m \times m$; $NA = 0$; and N nilpotent. Then AB and $AB + N$ have the same characteristic polynomials.

THEOREM 3. *The elementary divisors of $AB+N$ which do not have zero as a root coincide with those of AB.*

As above, we let $A \in \mathrm{Hom}\,(L, M)$, $B \in \mathrm{Hom}\,(M, L)$, $N \in \mathrm{End}\,(L)$,

[4] This proof was suggested by Professor H. F. Bohnenblust.

[5] *A note on the characteristic polynomials of certain matrices*, Proceedings of the American Mathematical Society vol. 1 (1950) pp. 584–585.

and select $p \geqq m$, $p \geqq n$. Let $C = AB + N$, $D = BA$. Then $L = LC^p \oplus \mathfrak{N}(C^p)$, $M = MD^p \oplus \mathfrak{N}(D^p)$. We shall prove that A induces an isomorphism A' on LC^p onto MD^p. Since for each j, $C^jA = AD^j$, we have $LC^pA = LAD^p \subset MD^p$. But $MD^p = MD^{p+1} = MB(AB)^pA = MBC^pA \subset LC^pA$. Thus A is on LC^p onto MD^p. We observe that for any r, $C^r = (AB)^r + (AB)^{r-1}N + \cdots + N^r$, hence for $r = 2p$, $C^{2p} = (AB)^{2p} + (AB)^{2p-1}N + \cdots + (AB)^{p+1}N^{p-1}$ since $N^p = N^m = 0$. If $xC^pA = 0$, then $x(AB)^pA = 0$, $x(AB)^{p+1} = x(AB)^{p+2} = \cdots = 0$. Thus $xC^{2p} = 0$, $xC^p = 0$, which proves that A' is an isomorphism. We finally have $CA' = A'D$ so that the contraction of C to LC^p is similar to the contraction of D to MD^p. Thus C and D have the same elementary divisors which do not have zero as a root. The theorem follows from Theorem 2.

CALIFORNIA INSTITUTE OF TECHNOLOGY

DIVISION ALGEBRAS OVER FIELDS OF FORMAL POWER SERIES

JOHN T. MOORE[1]

1. **Introduction.** By a field of formal power series we shall mean the field K of all formal power series in m variables $t_1, t_2, \cdots, t_m$ with coefficients in an algebraically closed field of characteristic zero. O. F. G. Schilling has shown that every algebraic extension of finite degree over K is an abelian extension, and the purpose of this note is that of using the result of Schilling to prove the following properties of division algebras over such fields.

THEOREM. *A central division algebra D over a formal power series field K in m variables is primary if and only if D is cyclic of prime power degree, and the exponent of D is then its degree. Every central division algebra D over K is then a direct product of cyclic algebras.*

2. **Properties of the coefficient field.** We shall be considering a field K which is maximally complete with respect to a valuation function V, with values in a discrete linearly ordered abelian group of

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[1] The results of this paper form a portion of a Ph.D. dissertation, written under the guidance of A. A. Albert. In the remainder of the thesis it is shown that certain division algebras of prime power degree which were constructed by R. Brauer are all primary (Tohôku Math. J. vol. 37 (1933) pp. 77–87).