GREEN'S SECOND IDENTITY FOR GENERALIZED LAPLACIANS

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Let $J(P, r)$ denote the closed circular disc bounded by the circle $C(P, r)$ with center at P, and radius r, in the plane. We define the generalized Laplacian of the function F at P by

$$\Lambda F(P) = \lim_{r\to 0} 4[m(F; P, r) - F(P)]/r^2,$$

where $m(F; P, r)$ is the mean value of F on $C(P, r)$. The upper and lower Laplacians $\Lambda^*F(P)$ and $\Lambda_*F(P)$ are defined likewise, with lim sup and lim inf in place of lim [3].[1] If $f\in L$ in a bounded domain R, we define [3, p. 281]

$$\Omega_R f(P) = -\frac{1}{2\pi}\iint_R f(Q)g(P, Q)dQ \qquad (P \text{ in } R),$$

where $g(P, Q)$ is Green's function for R. In [3] we established the existence of $\Lambda F(P)$ for almost all P of a domain D, the integrability of $\Lambda F(P)$ over any compact subset of D, and the formula

$$F(P) = \Omega_R \Lambda F(P) + H(P), \tag{1}$$

valid for almost all P of any bounded domain R such that $\overline{R}\subset D$, where H is harmonic in R and assumes the values of F on the boundary of R, under the following hypotheses:

(A) F is continuous in D;

(B) $\Lambda^*F(P) > -\infty$, $\Lambda_*F(P) < +\infty$, except possibly on a closed set of capacity zero;

(C) there exists a function y, defined in D, such that $y\in L$ on every compact subset of D, and such that $y(P) \leqq \Lambda^*F(P)$ for P in D.

In [4], (B) was slightly weakened. In the present paper the above result is used to obtain the following theorem.

THEOREM. *If the functions U and V satisfy* (A), (B), (C) *in a domain D, and if $U(P) = 0$ outside a compact subset K of D, then*

$$\iint_D U(P)\Lambda V(P)dP = \iint_D V(P)\Lambda U(P)dP. \tag{2}$$

Presented to the Society, February 24, 1951; received by the editors December 11, 1950

[1] Numbers in brackets refer to the references at the end of the paper.

If U and V have continuous second derivatives, then (2) is clearly an immediate consequence of Green's second identity [1, p. 215].

Let R be a bounded domain such that $K \subset R \subset \bar{R} \subset D$. Since U, and therefore ΛU, vanish outside K, it suffices to prove that (2) holds with R in place of D.

Putting $u(P) = \Lambda U(P)$, $v(P) = \Lambda V(P)$, wherever the Laplacians exist, we have, by (1), writing Ω for Ω_R,

$$U(P) = \Omega u(P), \qquad V(P) = \Omega v(P) + H(P) \qquad \text{(p.p. in } R). \tag{3}$$

By Fubini's theorem, and (3),

$$\begin{aligned}\iint_R v(P)\Omega u(P)dP &= \iint_R u(P)\Omega v(P)dP \\ &= \iint_R u(P)V(P)dP - \iint_R u(P)H(P)dP.\end{aligned}$$

Hence it is enough to prove that

$$\iint_R u(P)H(P)dP = 0 \tag{4}$$

for every function H harmonic in R. Choose a domain G such that $K \subset G \subset \bar{G} \subset R$. Choose $r > 0$ such that $J(P, 3r) \subset R$ if $P \in G$. Define $H(P) = 0$ outside R. Put $H_1(P) = A_r H(P)$ (that is, the mean of H on $J(P, r)$), $H_2(P) = A_r H_1(P)$, and $H_3(P) = A_r H_2(P)$, for all P. Then $H_3(P) = H(P)$ in G, H_3 has continuous second derivatives everywhere [2, p. 343], and $H_3(P) = 0$ outside some bounded domain T containing $\bar{G}$. Hence we have, for all P in T,

$$H_3(P) = \Omega_T h_3(P), \tag{5}$$

where $h_3(P) = \Lambda H_3(P)$. Noting that $u(P) = U(P) = 0$ wherever $H(P) \neq H_3(P)$, we obtain

$$\begin{aligned}\iint_R u(P)H(P)dP &= \iint_T u(P)\Omega_T h_3(P)dP \\ &= \iint_T h_3(P)\Omega_T u(P)dP = \iint_T U(P)\Lambda H(P)dP = 0,\end{aligned}$$

since $\Lambda H(P) = 0$ in K, and $U(P) = 0$ in $T - K$. This proves (4), and hence the theorem.

The extension to more than two dimensions is evident.

References

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2. F. Riesz, *Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel*, Acta Math. vol. 54 (1930) pp. 321–360.

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4. ———, *A theorem on subharmonic functions*, Proceedings of the American Mathematical Society vol. 2 (1951) pp. 209–212.

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