THE EXTENDED CENTRALIZER OF A RING OVER A MODULE

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In a recent paper,[1] K. Asano gave a new proof of the theorem that a domain of integrity has a right quotient ring if and only if every pair of nonzero elements has a common nonzero right multiple. His method of proof is used in the present work to extend the centralizer of a ring over a module to a system of semi-endomorphisms of the module. From this extension, necessary and sufficient conditions that a ring have a right quotient regular ring are derived.

Consider a given ring R, and a given nonzero right R-module M. Denote by $\mathfrak{M}$ the set of all submodules of M, and by $\mathfrak{M}^*$ the set of all submodules N of M having the property that $N \cap N' \neq 0$ for all nonzero $N' \in \mathfrak{M}$. Since $M \in \mathfrak{M}^*$, $\mathfrak{M}^*$ is not void. It is easily seen that if N and N' are in $\mathfrak{M}^*$, then $N+N'$ and $N \cap N'$ are also in $\mathfrak{M}^*$. Thus $\{\mathfrak{M}^*; \subseteq, \cap, +\}$ is a sublattice of the lattice $\{\mathfrak{M}; \subseteq, \cap, +\}$.

An R-homomorphism of N into M, N any element of $\mathfrak{M}$, is called a *semi-endomorphism* of M. Thus, thinking of the semi-endomorphism α as a left operator on N, we have $\alpha(x+y) = \alpha x + \alpha y$ and $\alpha(xa) = (\alpha x)a$ for all $x, y \in N$, $a \in R$. For convenience, the module N on which α is defined is denoted by M_α.

The set of all semi-endomorphisms of M is labeled with $\mathfrak{A}$. Contained in $\mathfrak{A}$ is the usual centralizer of R over M consisting of all $\alpha \in \mathfrak{A}$ for which $M_\alpha = M$. A partial ordering $\leqq$ is defined in $\mathfrak{A}$ as follows: $\alpha \leqq \beta$ if and only if $M_\alpha \subseteq M_\beta$ and $\alpha x = \beta x$ for all $x \in M_\alpha$. The notation $\alpha < \beta$ is used in case $\alpha \leqq \beta$ and $M_\alpha \neq M_\beta$.

In case $\mathfrak{L}$ is a linearly ordered subset of $\mathfrak{A}$, and $M' = \bigcup M_\alpha$, $\alpha \in \mathfrak{L}$, the mapping γ of M' into M defined by

$$\gamma x = \alpha x \quad \text{whenever} \quad x \in M_\alpha, \alpha \in \mathfrak{L},$$

is easily verified to be an element of $\mathfrak{A}$ such that $\gamma \geqq \alpha$ for all $\alpha \in \mathfrak{L}$. Thus, by Zorn's Lemma, every α of $\mathfrak{A}$ is contained in a maximal element of $\mathfrak{A}$. Let $\mathfrak{B}$ denote the set of all maximal elements of $\mathfrak{A}$. Obviously the centralizer of R over M is contained in $\mathfrak{B}$. For any $\beta \in \mathfrak{B}$, $M_\beta \in \mathfrak{M}^*$. Otherwise there would exist a nonzero $N \in \mathfrak{M}$ such that $N \cap M_\beta = 0$, and the semi-endomorphism α defined by

$$\alpha x = \beta x, \quad x \in M_\beta; \qquad \alpha x = 0, \quad x \in N,$$

would exceed β.

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[1] Journal of the Mathematical Society of Japan vol. 1 (1949) pp. 73–78.

For any $\alpha, \beta \in \mathfrak{A}$, define $M_\beta^\alpha \in \mathfrak{M}$ by

$$M_\beta^\alpha = \{x \mid x \in M_\beta, \beta x \in M_\alpha\}.$$

Observe that if $M_\alpha, M_\beta \in \mathfrak{M}^*$, then also $M_\beta^\alpha \in \mathfrak{M}^*$. For if $N \in \mathfrak{M}$, $N \neq 0$, then $N \cap M_\beta \neq 0$; if $\beta(N \cap M_\beta) \neq 0$, then $\beta(N \cap M_\beta) \cap M_\alpha \neq 0$, $(N \cap M_\beta) \cap M_\beta^\alpha \neq 0$, and therefore $N \cap M_\beta^\alpha \neq 0$; if, on the other hand, $\beta(N \cap M_\beta) = 0$, then $N \cap M_\beta \subseteq M_\beta^\alpha$ and again $N \cap M_\beta^\alpha \neq 0$.

Operations of addition and multiplication are defined in $\mathfrak{A}$ in the obvious way. Thus for $\alpha, \beta \in \mathfrak{A}$, $\alpha+\beta$ and $\alpha\beta$ are defined as follows:

$$(\alpha + \beta)x = \alpha x + \beta x, \ x \in M_\alpha \cap M_\beta; \quad (\alpha\beta)x = \alpha(\beta x), \ x \in M_\beta^\alpha.$$

By definition, $M_{\alpha+\beta} = M_\alpha \cap M_\beta$ and $M_{\alpha\beta} = M_\beta^\alpha$.

Associated with any $N \in \mathfrak{M}$ are the trivial semi-endomorphisms 0_N and 1_N defined by: $0_N x = 0$, $1_N x = x$; $x \in N$. Labelling $0_M = 0$ and $1_M = 1$, evidently $0_N \leqq 0$ and $1_N \leqq 1$ for all $N \in \mathfrak{M}$. For any $\alpha \in \mathfrak{A}$, $-\alpha$ is defined in the usual way; and $\alpha + (-\alpha) = (-\alpha) + \alpha = 0_N$ where $N = M_\alpha = M_{-\alpha}$.

Every $\alpha \in \mathfrak{A}$ that is an isomorphism of M_α into M has an inverse α^{-1} defined by

$$\alpha^{-1}(\alpha x) = x, \quad x \in M_\alpha; \qquad M_{\alpha^{-1}} = \alpha M_\alpha.$$

The set of all such isomorphisms contained in $\mathfrak{A}$ is denoted by $\mathfrak{U}$. It is evident that all $1_N \in \mathfrak{U}$, $N \in \mathfrak{M}$, and whenever $\alpha \in \mathfrak{U}$, also $\alpha^{-1} \in \mathfrak{U}$.

The properties enjoyed by the operations in $\mathfrak{A}$ are summarized in the following theorem.

THEOREM 1. *The algebraic system* $\{\mathfrak{A}; +, \cdot, \leqq\}$ *has the following properties*:

(1) $\{\mathfrak{A}; +\}$ *is an abelian semigroup with identity element* 0. *Associated with each* $\alpha \in \mathfrak{A}$ *are unique elements* $-\alpha$ *and* 0_α *in* $\mathfrak{A}$ *such that*

(i) $\alpha + (-\alpha) = 0_\alpha$, (ii) $\alpha + 0_\alpha = \alpha$, (iii) $-(-\alpha) = \alpha$.

(2) $\{\mathfrak{A}; \cdot\}$ *is a semigroup with identity element* 1.

(3) $\{\mathfrak{U}; \cdot\}$ *is a semigroup with identity element* 1. *Associated with each* $\alpha \in \mathfrak{U}$ *are unique elements* α^{-1}, 1_α, *and* $1'_\alpha$ *in* $\mathfrak{U}$ *such that*

(i) $\alpha^{-1}\alpha = 1_\alpha$, (ii) $\alpha\alpha^{-1} = 1'_\alpha$, (iii) $\alpha 1_\alpha = \alpha$,

(iv) $1'_\alpha \alpha = \alpha$, (v) $(\alpha^{-1})^{-1} = \alpha$.

(4) *For any* $\alpha, \beta, \gamma \in \mathfrak{A}$,

(i) $(\beta+\gamma)\alpha = \beta\alpha + \gamma\alpha$, (ii) $\alpha(\beta+\gamma) \geqq \alpha\beta + \alpha\gamma$.

(5) *For any* $\alpha, \beta, \gamma, \delta \in \mathfrak{A}$ *such that* $\alpha \leqq \beta$ *and* $\gamma \leqq \delta$,

(i) $\alpha + \gamma \leqq \beta + \delta$, (ii) $\alpha\gamma \leqq \beta\delta$, (iii) $-\alpha \leqq -\beta$,

(iv) $\alpha^{-1} \leqq \beta^{-1}$ *in case* $\alpha, \beta \in \mathfrak{U}$.

The proofs of (1)–(3) are straightforward, and hence will be omitted. Part (4) is a consequence of the following relations:

$$M_\alpha^{\beta+\gamma} = M_\alpha^\beta \cap M_\alpha^\gamma, \qquad M_{\beta+\gamma}^\alpha \supseteq M_\beta^\alpha \cap M_\gamma^\alpha.$$

To prove (5) part (iv), assume that $\alpha, \beta \in U$ with $\alpha < \beta$. Then $M_\alpha \subset M_\beta$, $M_\alpha \neq M_\beta$, so that $\alpha M_\alpha \subset \beta M_\beta$, $\alpha M_\alpha \neq \beta M_\beta$, and hence $\alpha^{-1} < \beta^{-1}$. The proof of the rest of (5) will be omitted.

Let $\mathfrak{K}$ be the subset of $\mathfrak{A}$ containing all α such that $M_\alpha \in \mathfrak{M}^*$. The set $\mathfrak{K}$ is closed under the operations of addition and multiplication in view of previous remarks. If in Theorem 1 we replace $\mathfrak{U}$ by $\mathfrak{V} = \{\alpha \mid \alpha, \alpha^{-1} \in \mathfrak{U} \cap \mathfrak{K}\}$, Theorem 1 then applies to the system $\{\mathfrak{K}; +, \cdot, \leqq\}$.

For any $\alpha \in \mathfrak{A}$, let $N_\alpha = \{x \mid x \in M_\alpha, \alpha x = 0\}$, the annihilator of α in M. The subset $\mathfrak{H}$ of $\mathfrak{K}$ defined by

$$\mathfrak{H} = \{\alpha \mid \alpha \in \mathfrak{K}, N_\alpha \in \mathfrak{M}^*\}$$

is called the *radical* of $\mathfrak{K}$. Since $N_{\alpha-\beta} \supseteq N_\alpha \cap N_\beta$ and $N_{\gamma\alpha} \supseteq N_\alpha$, $\alpha - \beta$ and $\gamma\alpha$ are in $\mathfrak{H}$ whenever $\alpha, \beta \in \mathfrak{H}$, $\gamma \in \mathfrak{K}$. That also $\alpha\gamma \in \mathfrak{H}$ for any $\alpha \in \mathfrak{H}$, $\gamma \in \mathfrak{K}$ is seen as follows. If $N \in \mathfrak{M}$, $N \neq 0$, then $N \cap M_\gamma \neq 0$. If $\gamma(N \cap M_\gamma) = 0$, then $N \cap M_\gamma \subseteq N_{\alpha\gamma}$ and $N \cap N_{\alpha\gamma} \neq 0$; while if $\gamma(N \cap M_\gamma) \neq 0$, then $\gamma(N \cap M_\gamma) \cap N_\alpha \neq 0$, $(N \cap M_\gamma) \cap N_{\alpha\gamma} \neq 0$, and again $N \cap N_{\alpha\gamma} \neq 0$. We conclude that the radical $\mathfrak{H}$ is an ideal in $\mathfrak{K}$. Another property of $\mathfrak{H}$ is that if $\beta \in \mathfrak{H}$ and $\alpha, \gamma \in \mathfrak{K}$ with $\alpha \leqq \beta \leqq \gamma$, then α and γ are in $\mathfrak{H}$. This is obvious, since $N_\alpha = M_\alpha \cap N_\beta$ and $N_\gamma \supseteq N_\beta$.

The ideal $\mathfrak{H}$ induces a partition of $\mathfrak{K}$ into the set $\mathfrak{K}/\mathfrak{H}$ of cosets $\bar{\alpha} = \{\beta \mid \beta \in \mathfrak{K}, \alpha - \beta \in \mathfrak{H}\}$, $\alpha \in \mathfrak{K}$. The operations of addition and multiplication can be introduced into $\mathfrak{K}/\mathfrak{H}$ in the usual way, that is, if $\alpha + \beta = \gamma$ and $\alpha\beta = \delta$, then $\bar{\alpha} + \bar{\beta} = \bar{\gamma}$ and $\bar{\alpha}\bar{\beta} = \bar{\delta}$. One easily verifies that $\{\mathfrak{K}/\mathfrak{H}; +\}$ is an abelian group. By Theorem 1, (4), we see that $(\bar{\beta} + \bar{\gamma})\bar{\alpha} = \bar{\beta}\bar{\alpha} + \bar{\gamma}\bar{\alpha}$ and $\bar{\alpha}(\bar{\beta} + \bar{\gamma}) = \bar{\alpha}\bar{\beta} + \bar{\alpha}\bar{\gamma}$. Thus $\{\mathfrak{K}/\mathfrak{H}; +, \cdot\}$ is a ring with a unit element. This ring is called the *extended centralizer* of R over M.

THEOREM 2. *The extended centralizer of R over M is a regular ring.*

To prove this, it is necessary to show that for every $\bar{\alpha} \in \mathfrak{K}/\mathfrak{H}$, there exists $\bar{\beta} \in \mathfrak{K}/\mathfrak{H}$ such that $\bar{\alpha}\bar{\beta}\bar{\alpha} = \bar{\alpha}$. Let us assume that $\bar{\alpha} \neq 0$, from which we can deduce that $N_\alpha \notin \mathfrak{M}^*$. Consequently there exists a maximal element $N \in \mathfrak{M}$ such that $N \neq 0$, $N_\alpha \cap N = 0$, and $N \subseteq M_\alpha$. The maximality of N implies that $N_\alpha + N \in \mathfrak{M}^*$. Now α is an isomorphism of N into M, so there exists some $\beta \in \mathfrak{K}$ such that $\beta(\alpha x) = x$, $x \in N$. Evidently $(\alpha\beta\alpha)x = \alpha x$, $x \in N$; $(\alpha\beta\alpha)x = 0$, $x \in N_\alpha$, so that $\alpha\beta\alpha - \alpha \in \mathfrak{H}$. Thus $\bar{\alpha}\bar{\beta}\bar{\alpha} = \bar{\alpha}$ and the theorem follows.

The set of all submodules of M other than 0 is denoted by $\mathfrak{M} - (0)$.

COROLLARY. *The extended centralizer of R over M is a division ring if and only if* $\mathfrak{M}^* = \mathfrak{M} - (0)$.

If $N \cap N' \neq 0$ for every pair of nonzero submodules of M, then the radical $\mathfrak{H}$ and $\mathfrak{K}$ consists of the non-isomorphisms of $\mathfrak{K}$. If $\bar{\alpha}\bar{\beta} = 0$, $\bar{\alpha}$, $\bar{\beta} \in \mathfrak{K}/\mathfrak{H}$, then either α or β is a non-isomorphism (since the product of isomorphisms is an isomorphism), that is, either $\bar{\alpha} = 0$ or $\bar{\beta} = 0$. Thus $\mathfrak{K}/\mathfrak{H}$ is a domain of integrity, and hence a division ring.

On the other hand, if $N \cap N' = 0$ for some pair of nonzero submodules of M, no loss of generality results from assuming that $N + N' \in \mathfrak{M}^*$. Then it is possible to find elements α, $\beta \in \mathfrak{K}$ with $N_\alpha = N$, $N_\beta = N'$, and $\alpha x = x$, $x \in N'$, $\beta x = x$, $x \in N$. Obviously $\bar{\alpha}$, $\bar{\beta} \neq 0$, while $\bar{\alpha}\bar{\beta} = 0$. This proves the corollary to Theorem 2.

In the ring R, let $I_a = \{x \mid x \in R,\ ax = 0\}$, the right annihilator of the element a of R. The element a of R is called (right) *singular* in case $I_a \cap I \neq 0$ for all nonzero right ideals I of R. The set S of all (right) singular elements of R is shown to be an ideal in much the same way that $\mathfrak{H}$ is shown to be an ideal in $\mathfrak{K}$. We shall call S the (right) *singular ideal* of R.

If the ring R is a subring of the ring Q, Q is called a (right) *quotient ring* of R if Q has a unit element, and for every $\alpha \in Q$, $\alpha \neq 0$, there exist elements a, b in R with $b \neq 0$ such that $\alpha a = b$. If, in addition, Q is a regular ring, then Q is called a regular quotient ring of R.

Let us now show that a ring having a regular quotient ring Q has its singular ideal equal to zero. For every $\beta \in Q$, we denote by I_β the right annihilator of β in R; thus I_β is a right ideal in R. If $a \in R$, $a \neq 0$, there exists $\alpha \in Q$ such that $a\alpha a = a$. Evidently $\epsilon = \alpha a$ and $1 - \epsilon$ are idempotents, and $I_\epsilon = I_a$. If $I_a \neq 0$, $\epsilon \neq 1$ and $1 - \epsilon \neq 0$. By assumption, there exist c, $d \in R$ with $d \neq 0$ such that $\epsilon c = d$. Since $d \in I_{1-\epsilon}$, both I_ϵ and $I_{1-\epsilon}$ are unequal to zero. Certainly $I_\epsilon \cap I_{1-\epsilon} = 0$, and therefore a is not singular in R. Note that if $a \in R$ and $I_a = 0$, then $\alpha a = 1$; that is, the elements of R having no nonzero right annihilator have left inverses in Q.

A possible choice of a right R-module for any ring R is the additive group R^+ of R. Then the left multiplications of R, that is, the mappings a' of R^+ defined by $a'x=ax$, $x\in R^+$, $a\in R$, are in the centralizer of R over R^+. If α is a semi-endomorphism of R^+, and $\alpha x=y$ for some $x, y\in R^+$, then it follows easily that $\alpha x'=y'$.

We now assume that the (right) singular ideal of R is zero, and that $M=R^+$. Then for any $a\in R$, $a\neq 0$, $N_{a'}\in\mathfrak{M}$, and therefore $a'\notin\mathfrak{H}$, the radical of $\mathfrak{K}$. As a matter of fact, if $\alpha\in\mathfrak{K}$, $\alpha M_\alpha\neq 0$, then $\alpha x=y\neq 0$ for some $x, y\in M$, and $\alpha x'=y'\notin\mathfrak{H}$ so that $\alpha\notin\mathfrak{H}$. It follows that the radical of $\mathfrak{K}$ consists of all 0_N, $N\in\mathfrak{M}^*$, and that the elements of the extended centralizer Q are essentially the maximal semi-endomorphisms (since $\alpha\leqq\beta$ implies $\bar{\alpha}=\bar{\beta}$) of M. Thus R is (isomorphic to) a subring of Q, and in view of Theorem 2, Q is a regular quotient ring of R. We have proved the following theorem.[2]

THEOREM 3. *A ring R has a (right) regular quotient ring if and only if the (right) singular ideal of R is zero.*

The extended centralizer Q of R over R^+ is the universal quotient ring of R in the sense that any quotient ring P of R is a subring of Q. For if $\alpha\in P$, $\alpha\neq 0$, α can be thought of as a semi-endomorphism of R^+ with $M_\alpha=I^+$ where $I=\{a\mid a\in R,\ \alpha a\in R\}$. Since $\alpha M_\alpha\neq 0$, P is a subring of Q.

In case R is a domain of integrity, its singular ideal is zero, and therefore R has a regular quotient ring Q. By the corollary of Theorem 2, Q is a division ring if and only if $\mathfrak{M}^*=\mathfrak{M}-(0)$, that is, if and only if $xR\cap yR\neq 0$ for every pair of nonzero elements $x, y\in R$. This yields the following corollary.

COROLLARY. *Any domain of integrity R has a (right) regular quotient ring Q, and each nonzero element of R has a left inverse in Q. The ring Q is a division ring if and only if every pair of nonzero elements of R has a common nonzero right multiple.*

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[2] O. Goldman, Bull. Amer. Math. Soc. vol. 52 (1946) p. 130, gives necessary and sufficient conditions that a ring be a subring of a regular ring.