PATTERN INTEGRATION WITH IMPROPER Riemann Integrals

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1. Introduction. In a previous paper \([1]\)^1 we have recently introduced and discussed a type of integration, which we have called pattern integration, and have established the following theorem:

**Theorem 1.1.** Let \(f(x)\) be (proper) Riemann integrable, \(0 \leq x \leq 1\). Let \((k-1)/n \leq \xi_k^{(n)} \leq k/n\) \((k = 1, 2, 3, \cdots, n)\). Let the pattern \(P\) be characterized by the dyadic number

\[ t = 0.a_1a_2a_3 \cdots a_n \cdots (2), \]

such that

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} a_k = \alpha. \]

Then the pattern integral

\[ (P) \int_{0}^{1} f(x)dx \equiv \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \alpha_k f(\xi_k^{(n)}) = \alpha(R) \int_{0}^{1} f(x)dx. \]

It seems desirable to extend this result, if possible, to improper Riemann integrals. In this paper we shall confine our remarks to improper integrals of the form

\[ \int_{0}^{1} f(x)dx \equiv \lim_{\varepsilon \to 0^+} (R) \int_{\varepsilon}^{1} f(x)dx. \]

\((R)\) in front of any integral will indicate a proper Riemann integral.

2. The improper integral defined as the limit of a sum. We first consider the case where \(\alpha_k = 1\) \((k = 1, 2, 3, \cdots)\). A fundamental difficulty not encountered in the previous paper is apparent from the definition of \(\int_{0}^{1} f(x)dx\). Bromwich and Hardy have considered this problem \([2]\). As they point out, the ordinary definition of this improper integral is by means of a double (repeated) limit; and in order to replace this limit by the single limit,

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(\xi_k^{(n)}), \]

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^1 Numbers in brackets refer to the bibliography at the end of the paper.
some restrictions are certainly necessary. In the first place, if $0 < \xi_1^{(n)} \leq 1/n$, we can clearly choose $\xi_1^{(n)}$ so that $|(1/n)f(\xi_1^{(n)})|$ is greater than any assigned number. It is then suggested that the most obvious choice is $\xi_2^{(n)} = k/n$, and with this restriction they establish:

**Theorem 2.1.** Let $f(x)$ be positive and tend steadily to $\infty$ as $x$ tends to zero. Let $\int_0^1 f(x)dx$ be convergent. Then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f \left( \frac{k}{n} \right) = \int_0^1 f(x)dx.$$

The proof rests on the fact that for all $n$,

$$\frac{1}{n} \sum_{k=2}^{n} f \left( \frac{k}{n} \right) \leq \int_{1/n}^1 f(x)dx \leq \frac{1}{n} \sum_{k=1}^{n-1} f \left( \frac{k}{n} \right),$$

while the difference between the two sums in (2.1) is

$$\frac{1}{n} \left( f \left( \frac{1}{n} \right) - f(1) \right).$$

That the limit of (2.2), as $n \to \infty$, is zero, follows from the well known result that if $f(x)$ steadily increases as $x$ decreases and $\int_0^1 f(x)dx$ is convergent, then $\lim_{x \to 0^+} x f(x) = 0$. From this the conclusion is immediate.

It appears that the following extension of the above theorem is new, although the theorem is quite elementary.

**Theorem 2.2.** Let $f(x)$ be positive and tend steadily to $\infty$ as $x$ tends to zero. Let $\int_0^1 f(x)dx$ be convergent. Let $(1/Mn) \leq \xi_1^{(n)} \leq (1/n)$ ($M \geq 1$ and fixed), and $(k-1)/n \leq \xi_k^{(n)} \leq (k/n)$ ($k = 2, 3, \ldots, n$). Then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(\xi_k^{(n)}) = \int_0^1 f(x)dx.$$

**Proof.** Define

$$A_n = \frac{1}{n} \sum_{k=1}^{n} f \left( \frac{k}{n} \right)$$

and

$$B_n = \frac{1}{n} \left( f \left( \frac{1}{Mn} \right) + \sum_{k=2}^{n} f \left( \frac{k-1}{n} \right) \right)$$

$$= \frac{1}{n} \left( f \left( \frac{1}{Mn} \right) + \sum_{k=1}^{n} f \left( \frac{k}{n} \right) - f(1) \right).$$
Then
\[ A_n = -\sum_{k=1}^{n} f(\xi_k^{(n)}) \leq B_n. \]

But, clearly, \( \lim_{n \to \infty} A_n = \lim_{n \to \infty} B_n = \int_0^1 f(x) \, dx \), and our conclusion follows.

A sharper criterion, in order that the relation
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f \left( \frac{k}{n} \right) = \int_0^1 f(x) \, dx \]
should hold whenever the improper integral on the right is convergent, has been given by Wintner [3] in the following result:

**Theorem 2.3.** If \( f(x) \) is of bounded variation on every interval \( \epsilon \leq x \leq 1 \), where \( \epsilon > 0 \), and behaves, as \( x \to 0 \), so as to satisfy the restriction
\[ \int_{\epsilon}^1 |f(x)| = o(\epsilon^{-1}), \]
then the convergence of \( \int_0^1 f(x) \, dx \) implies that
\[ \lim_{\epsilon \to 0^+} \epsilon \sum_{k \leq 1} f(ke) \]
exists and equals \( \int_0^1 f(x) \, dx \).

The result (2.3) would follow from (2.4) by letting \( \epsilon = 1/n \).

The following extension of Wintner's result appears to be new:

**Theorem 2.4.** Let \( f(x) \) be of bounded variation on every interval \( \epsilon \leq x \leq 1 \), where \( \epsilon > 0 \), and let \( f(x) \) behave, as \( x \to 0^+ \), so as to satisfy the restriction
\[ \int_{\epsilon}^1 |f(x)| = o(\epsilon^{-1}). \]

Let \( \xi_1 = 1/Mn \) (\( M \geq 1 \) and fixed), and \( (k-1)/n \leq \xi_k \leq (k/n) \) \( (k = 2, 3, \ldots, n) \). Let \( \int_0^1 f(x) \, dx \) exist.

Then
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(\xi_k^{(n)}) = \int_0^1 f(x) \, dx. \]

**Proof.** From the hypothesis it follows that
\[ \lim_{x \to 0} xf(x) = 0. \]
For,
\[ \epsilon |f(1) - f(\epsilon)| = \epsilon \left| \int_{\epsilon}^{1} df(x) \right| \leq \epsilon \int_{\epsilon}^{1} |df(x)| \rightarrow 0. \]

From Wintner's result, we know
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_{0}^{1} f(x)dx. \]

Consider
\[ \left\{ \frac{1}{n} \sum_{k=1}^{n} f(\xi_{k}^{(n)}) - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right\}. \]

For all \( n \),
\[ \left| \sum_{k=1}^{n} f(\xi_{k}^{(n)}) - \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right| \leq \frac{n}{\sum_{k=1}^{n}} \left| f(\xi_{k}^{(n)}) - f\left(\frac{k}{n}\right) \right| \]
\[ < \int_{(1/n)}^{1} |df(x)| + \left| f\left(\frac{1}{Mn}\right) \right| + \left| f\left(\frac{1}{n}\right) \right| . \]

Hence, using (2.5) and (2.6),
\[ \lim_{n \to \infty} \left| \frac{1}{n} \sum_{k=1}^{n} f(\xi_{k}^{(n)}) - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right| = 0, \]

and our conclusion follows.

It should be noted that a number of theorems are available in the "converse" direction, that is, in which the convergence of the improper integral \( \mathcal{I}_{1}f(x)dx \) follows from the knowledge of the existence of \( \lim_{n \to \infty} \sum_{n=1}^{N} f(n\epsilon) \). In this paper, however, we are not concerned with such results.

3. A pattern integral theorem. To hope for a complete extension of Theorem 1.1 to improper integrals of the form \( \mathcal{I}_{1}f(x)dx \) is too much. Counter examples are not difficult to construct. For example, see §5 of the paper by Bromwich and Hardy [2].

The following theorem partially extends the earlier results.

**Theorem 3.1.** Let \( f(x) \) be positive and tend steadily to \( \infty \) as \( x \) tends to zero. Let \( \mathcal{I}_{1}f(x)dx \) be convergent. Let \( (1/Mn) \leq \xi_{1}^{(n)} \leq (1/n) \) \((M \geq 1 and fixed)\), and \((k-1)/n \leq \xi_{k}^{(n)} \leq k/n \((k = 2, 3, \cdots, n)\). Let the pattern
\( P \) be characterized by the dyadic number
\[
t = 0.\alpha_1\alpha_2\alpha_3 \cdots \alpha_n \cdots \tag{2},
\]
such that
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \alpha_k = \alpha.
\]

Then
\[
\left( P \right) \int_{0}^{1} f(x) \, dx = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \alpha_k f(\xi_k^{(n)}) = \alpha \int_{0}^{1} f(x) \, dx.
\]

**Proof.** Define \( f_N(x) = f(1/MN) \), \( 0 \leq x < 1/N \), \( f_N(x) = f(x) \), \( 1/N \leq x \leq 1 \). For all values of \( N > 0 \), \( (R) \int_{0}^{1} f_N(x) \, dx \) exists and
\[
(3.1) \quad \left( P \right) \int_{0}^{1} f_N(x) \, dx = \alpha(R) \int_{0}^{1} f_N(x) \, dx.
\]

Consider the expression
\[
(3.2) \quad \frac{1}{n} \sum_{k=1}^{n} \alpha_k f(\xi_k^{(n)}).
\]

Let the subinterval in which \( 1/N \) occurs be the \( \lambda(n) \)th. (Then \( \lim_{n \to \infty} \lambda(n)/n = 1/N \).) Hence (3.2) can be written as \( I_n + \Pi_n - \PiI_n + IV_n \), where
\[
I_n = \frac{1}{n} \alpha_{\lambda(n)} f(\xi_{\lambda(n)}^{(n)}), \quad \Pi_n = \frac{1}{n} \sum_{k=1}^{\lambda(n)-1} \alpha_k f_N(\xi_k^{(n)}),
\]
\[
\PiI_n = \frac{1}{n} \sum_{k=1}^{\lambda(n)} \alpha_k f_N(\xi_k^{(n)}), \quad IV_n = \frac{1}{n} \sum_{k=1}^{\lambda(n)-1} \alpha_k f(\xi_k^{(n)}).
\]

Let us consider the behavior of \( I_n, \Pi_n, \PiI_n, \) and \( IV_n \), as \( n \to \infty \). Clearly \( f(\xi_{\lambda(n)}^{(n)}) \leq f(1/Mn) \), and since \( \lim_{x \to 0^+} xf(x) = 0 \), \( \lim_{n \to \infty} I_n = 0 \). From (3.1), we see that
\[
\lim_{n \to \infty} \Pi_n = \alpha(R) \int_{0}^{1} f_N(x) \, dx = \alpha \left\{ \frac{1}{N} f \left( \frac{1}{MN} \right) + (R) \int_{1/N}^{1} f(x) \, dx \right\}.
\]

Now
\[
\PiI_n = f \left( \frac{1}{MN} \right) \frac{\lambda(n) - 1}{n} \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)-1} \alpha_k + \frac{1}{n} \alpha_{\lambda(n)} f_N(\xi_{\lambda(n)}^{(n)}).
\]

Hence,
\[
\lim_{n \to \infty} III_n = \frac{1}{N} f \left( \frac{1}{MN} \right).
\]

We have

\[
IV_n \leq \frac{1}{n} \sum_{k=1}^{\lambda(n)-1} f(\xi_k^{(n)}).
\]

Hence

\[
0 \leq IV_n \leq \int_0^{1/N} f(x) \, dx + \frac{1}{n} f \left( \frac{1}{Mn} \right)
\]

for all \( n \). We then see that, as \( n \to \infty \), the expression (3.2) at worst oscillates between

\[\alpha(R) \int_{1/N}^1 f(x) \, dx\]

and

\[\alpha(R) \int_{1/N}^1 f(x) \, dx + \int_0^{1/N} f(x) \, dx.\]

But this is true for all \( N > 0 \), so letting \( N \to \infty \), we see from (3.3) and (3.4) that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \alpha_k f(\xi_k^{(n)}) = \alpha(R) \int_0^1 f(x) \, dx,
\]

which is the desired result.

4. Miscellaneous results. The following is an extension of Theorem 3.1.

THEOREM 4.1. Let \( \mathcal{F}_0^1 f(x) \, dx \) be convergent. Let there be a \( g(x) \) with the following properties:

1. In the interval \( 0 < x \leq r \), where \( 0 < r \leq 1 \), \( g(x) \) is positive and tends steadily to \( \infty \) as \( x \) tends to zero.

2. \( \mathcal{F}_0^r g(x) \, dx \) is convergent.

Let \( |f(x)| \leq g(x), 0 < x \leq r \). Let \( 1/Mn \leq \xi_k^{(n)} \leq 1/n \) \( (M \geq 1 \) and fixed), and \( (k-1)/n \leq \xi_k^{(n)} \leq k/n \) \( (k = 2, 3, \ldots, n) \). Let the pattern \( P \) be characterized by the dyadic number

\[ t = 0.\alpha_1\alpha_2\alpha_3 \cdots \alpha_n \cdots (2), \]

such that
Then,

\[
(P) \int_0^1 f(x) \, dx = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \alpha_k f(\xi_k) = \alpha \int_0^1 f(x) \, dx.
\]

Since the proof is similar to that of Theorem 3.1, it will be omitted.

It is of interest to note that if one chooses

\[
f(x) = \frac{1}{x} \cos \frac{1}{x},
\]

then \( \int_0^1 f(x) \, dx \) converges, but

\[
(4.1) \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(\xi_k) = \alpha \int_0^1 f(x) \, dx.
\]

fails to exist.

Define

\[
A_n = \frac{1}{n} f\left(\frac{1}{Mn}\right) + \frac{1}{n} \sum_{k=2}^{n} f\left(\frac{k-1}{n}\right) \quad (M \geq 1 \text{ and fixed}),
\]

\[
B_n = \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right).
\]

Then \( A_n - B_n = (1/n) \left\{ f(1/Mn) - f(1) \right\} \). If (4.1) exists,

\[
\lim_{n \to \infty} \frac{M}{Mn} f\left(\frac{1}{Mn}\right) = 0,
\]

which is certainly not the case when \( f(x) = (1/x) \cos (1/x) \).

**Bibliography**

