1. Introduction. It is the purpose of this paper to obtain a formula for $\delta G$ when $G$ is the indefinite Wiener integral

$$G(u) = \int_{{x(t) \leq u(t)}}^W F(x) \, dw(x).$$

Here $x$ is understood to be a variable point in the Wiener space $C$ of continuous functions $x(t)$ defined on $0 \leq t \leq 1$ and vanishing at $t=0$. The integration is performed over the set $S_u$ of elements $x(t)$ of $C$ which satisfy for all $t$ the inequality

$$x(t) \leq u(t).$$

The function $u(t)$ need not be a member of $C$, but can be any Borel measurable function defined on $0 \leq t \leq 1$, and may even be permitted to take on infinite values.

The Wiener integral of a functional is simply the Lebesgue integral of the functional with respect to Wiener's measure $[4]^1$ in $C$. This measure is not invariant under translations, but is in other respects a Lebesgue measure based on intervals of the form

$I: \quad \alpha_j < x(t_j) < \beta_j$ (where $0 < t_1 < t_2 < \cdots < t_n \leq 1$),

having the measure

$$m_W(I) = \frac{1}{(\pi t_1(t_2 - t_1) \cdots (t_n - t_{n-1}))^{1/2}} \int_{\alpha_n}^{\beta_n} \cdots \int_{\alpha_1}^{\beta_1} \exp \left\{ \frac{-\xi_1^2}{t_1} \right\} \cdot \cdots \cdot \exp \left\{ \frac{-(\xi_n - \xi_{n-1})^2}{t_n - t_{n-1}} \right\} \, d\xi_1 \cdots d\xi_n.$$

We denote the Wiener integral of a measurable functional $F(x)$ over a measurable set $S \subset C$ by

$$\int_{S}^W F(x) \, dw(x).$$

In case $S$ is not contained in $C$ but $SC$ is measurable, we define

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1 Numbers in brackets refer to the bibliography at the end of the paper.

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\[ m_w(S) = m_w(SC) \text{ and } \int_S^W F(x) dw x = \int_{SC}^W F(x) dw x. \]

In addition to finding \( \delta G \) when \( G \) is given by (1), we shall also find certain transformation formulas for Wiener integrals taken over the whole of \( C \). In particular, we shall obtain what may be considered as a formula for integration by parts in function space.

Finally, we show that these formulas may be used to evaluate certain Wiener integrals. As an example we show that

\[ \int_C^W \left[ \log \int_0^1 \alpha(t)e^{x(t)dt} \right] x(s) dw x = \frac{s}{2} \]

when \( 0 < s < 1 \), if \( \alpha(t) \) is non-negative and of class \( L_1 \) and not equivalent to zero on \( s \leq t \leq 1 \).

2. Sub-summable functionals.

Definitions. Let \( f(u) \) be a real or complex function defined on a set \( S \) of an abstract space in which a measure is defined. Then \( f(u) \) will be called "sub-summable" on \( S \) if there exists a function \( g(u) \) which is summable on a measurable set \( S \) containing \( S \) and satisfies on \( S \) the inequality \( |f(u)| \leq g(u) \).

It is clear that if \( f(u) \) is also measurable on \( S \), then it is summable on \( S \) (and, of course, \( S \) is also measurable).

Lemma. Let \( y_0(t) \in C \) be absolutely continuous and have a derivative \( y'_0(t) \) which is essentially of bounded variation\(^2\) on \( [0, 1] \); let \( S \) be a Wiener measurable subset of \( C \); and for each positive \( \lambda \) let \( S^{\lambda} \) be the set of all \( x(t) \) of the form \( u(t) + hy_0(t) \), where \( u \in S \) and \( |h| \leq \lambda \). Let \( \epsilon > 0 \), \( \eta > 0 \) and let \( F(x) \) be a functional defined on \( S^{\epsilon+\lambda} \) such that

\[ \sup_{|h| \leq \eta} |F(x + hy_0)| \]

is sub-summable on \( S^\epsilon \). Then if \( P(\omega) \) is any polynomial, it follows that there exists \( \eta_1 > 0 \) for which

\[ \sup_{|h| \leq \eta_1} |F(x + hy_0)| \]

\[ \cdot \exp \left\{ 2\eta_1 \left| \int_0^1 y'_0(t) dx(t) \right| \right\} P \left( \int_0^1 y'_0(t) dx(t) \right) \]

is sub-summable on \( S \).

\(^2\) Here and elsewhere in this paper the requirement that a function be "essentially of bounded variation" can be replaced by the requirement that it be "of class \( L_t \)" if Stieltjes integrals are interpreted as Paley-Wiener-Zygmund integrals [3, 1].
For the proof, choose $\eta_2 > 0$ so that $\eta_2 < \epsilon$ and $\eta_2 < \eta/2$, and apply the translation theorem [2; 1] to a functional $G(x)$ which is summable on a measurable set containing $S^\epsilon$ and which satisfies on $S^\epsilon$

\[(5)\quad G(x) \geq \sup_{|h| \leq \eta} |F(x + hy_0)|.
\]

We translate by $\eta_2 y_0$ and also by $-\eta_2 y_0$, and obtain

\[(6)\quad \int_{(x \mp \eta_2 y_0) \in S} G(x)dx = \exp \left\{ -\frac{1}{\eta_2} \int_0^1 [y_0'(t)]^2 dx(t) \right\}
\cdot \int_S G(x \pm \eta_2 y_0) \exp \left\{ \mp 2\eta_2 \int_0^1 y_0'(t)dx(t) \right\} dwx.
\]

Here the existence of the first member follows from the fact that we are integrating over a measurable subset of $S^\epsilon$, and the existence of the second follows from that of the first by the translation theorem. Moreover, since $2\eta_2 < \eta$, we have by (5) for $x \in S$ and for both upper and lower signs,

\[\sup_{|h| \leq \eta_2} |F(x + hy_0)| \leq G(x \pm \eta_2 y_0),\]

and hence it follows from the existence of the Wiener integral in the second member of (6) that

\[\sup_{|h| \leq \eta_2} |F(x + hy_0)| \exp \left\{ \pm 2\eta_2 \int_0^1 y_0'(t)dx(t) \right\}\]

is sub-summable on $S$. Thus the maximum of these two functionals is also sub-summable; that is,

\[\sup_{|h| \leq \eta_2} |F(x + hy_0)| \exp \left\{ 2\eta_2 \int_0^1 y_0'(t)dx(t) \right\}\]

is sub-summable on $S$. The sub-summability of (4) on $S$ follows immediately for positive $\eta_1 < \eta_2$, and hence the lemma is established.

3. The first variation of a Wiener integral and vice versa.

**Theorem I.** Let $y_0(t) \in C$ be absolutely continuous and have a derivative $y_0'(t)$ which is essentially of bounded variation on $0 \leq t \leq 1$, and let $F(x)$ be a Wiener summable functional over the set $S_{u_0 + 2\epsilon}$, where $u_0(t)$ is Borel measurable on $0 \leq t \leq 1$ (and may even be infinite there), $\epsilon > 0$, and $S_{u^}:

\[x(t) \leq u(t), \quad 0 \leq t \leq 1; x \in C.\]
Let $F(x)$ have a first variation

\begin{equation}
\delta F \equiv \delta F(x \mid y_0) = \frac{d}{dh} F(x + hy_0) \bigg|_{h=0}
\end{equation}

for all $x \in S_{u+2\varepsilon}$. Then if $0 < \eta \max_{0 \leq t \leq 1} \mid y_0(t) \mid \leq \epsilon$ and

\begin{equation}
\sup_{\|h\| \leq \eta} \mid \delta F(x + hy_0 \mid y_0) \mid
\end{equation}

is Wiener sub-summable in $x$ on $S_{u+\varepsilon}$, it follows that the functional

\begin{equation}
G(u) = \int_{x(t) \leq u(t)} W(x)dx
\end{equation}

has a first variation

\begin{equation}
\delta G = \delta G(u \mid y_0) = \frac{d}{dh} G(u + hy_0) \bigg|_{h=0}
\end{equation}

whenever $u(t) \leq u_0(t)$ on $0 \leq t \leq 1$ and $u$ is Borel measurable. Moreover the value of the variation is given by the following integrals (which necessarily exist):

\begin{equation}
\delta G(u \mid y_0) = \int_{x(t) \leq u(t)} \delta F(x \mid y_0)dx
\end{equation}

\begin{equation}
-2 \int_{x(t) \leq u(t)} F(x) \left[ \int_{0}^{1} y_0(t)dx(t) \right] dx
\end{equation}

For the proof, we note that if $x + hy_0 \in S_{u+2\varepsilon}$, $\delta F(x + hy_0 \mid y_0) = \frac{d}{d\lambda} F(x + hy_0 + \lambda y_0) \bigg|_{\lambda=0}$

\begin{equation}
= \frac{d}{d\mu} F(x + \mu y_0) \bigg|_{\mu=h} = \frac{d}{dh} F(x + hy_0);
\end{equation}

and since the first member of this equation exists, so does the last.

Again, it is clear that $S_{u+2\varepsilon}$ is convex, so that if $x \in S_{u+2\varepsilon}$ and $x + hy_0 \in S_{u+2\varepsilon}$, we have $x + \theta hy_0 \in S_{u+2\varepsilon}$ for all $\theta$ in $0 \leq \theta \leq 1$. Thus by the law of the mean we obtain $F(x + hy_0) = F(x) + h\delta F(x + \theta hy_0 \mid y_0)$ for some $\theta$ in $0 < \theta < 1$ depending on $h$. Hence it follows from the sub-summability of (8) and of $F(x)$ that

\begin{equation}
\sup_{\|h\| \leq \eta} \mid F(x + hy_0) \mid
\end{equation}

is sub-summable on $S_{u+\varepsilon}$. 
Now for $|h| \leq \eta$ and $u$ a Borel measurable function satisfying $u(t) \leq u_0(t)$, we have by the translation theorem (which guarantees the existence of the last member)

$$G(u + hy_0) = \int_{x(t) - hy_0(t) \in S_u} F(x)dwx$$

$$= \exp \left\{ - h^2 \int_0^1 [y'_0(t)]^2 dt \right\} \int_{S_u} F(x + hy_0)$$

$$\cdot \exp \left[ - 2h \int_0^1 y'_0(t)dx(t) \right] dwx.$$ 

Differentiating formally with respect to $h$ and then setting $h = 0$, we obtain

$$\delta G(u | y_0) = \frac{d}{dh} G(u + hy_0) \bigg|_{h=0}$$

$$= \int_{S_u} \left[ \frac{d}{dh} \left\{ F(x + hy_0) \right\} \cdot \exp \left[ - 2h \int_0^1 y'_0(t)dx(t) \right] \right] dwx$$

$$= \int_{S_u} \delta F(x | y_0) dwx$$

$$- 2 \int_{S_u} F(x) \left[ \int_0^1 y'_0(t)dx(t) \right] dwx.$$

To justify this differentiation under the integral sign (and incidentally show that all members of (13) exist), we must show that the differentiated integrand is dominated for small $h$ by a summable functional; that is, we must show that

$$\sup_{|h| \leq \eta_1} \left\{ \delta F(x + hy_0 | y_0) - 2F(x + hy_0) \int_0^1 y'_0(t)dx(t) \right\}$$

$$\cdot \exp \left\{ - 2h \int_0^1 y'_0(t)dx(t) \right\}$$

is sub-summable on $S_u$ for some $\eta_1 > 0$. But it follows from the sub-summability of (8) on $S_{u_0+\epsilon}$ and the lemma that for some $\eta_2 > 0$

$$\sup_{|h| \leq \eta_2} |\delta F(x + hy_0 | y_0)| \exp \left\{ 2\eta_2 \int_0^1 y'_0(t)dx(t) \right\}$$
is sub-summable on $S_{u^p}$. Similarly it follows from the sub-summability of (12) on $S_{u^p}$ and the lemma that for some $\eta_2>0$,

$$\sup_{|h| \leq \eta_2} |F(x + hy_0) - \exp \left\{ 2\eta_2 \int_0^1 y_0'(t) dx(t) \right\} | \int_0^1 y_0'(t) dx(t)$$

is sub-summable on $S_{u^p}$. Taking $\eta_1 = \min (\eta_2, \eta_3)$, we obtain the sub-summability of (14) on $S_{u^p}$ and hence the justification of (13), including the existence of all its members. Thus the theorem is established.

An important special case of Theorem 1 is obtained if $u(t) = u_0(t) = +\infty$, so that we integrate over the whole space $C$. In this case also $u(t) + hy_0(t) = +\infty$ and $G(u + hy_0)$ is constant and $\delta G(u | y_0) = 0$. We state the result as a separate theorem.

**Theorem II.** Let $y_0(t)$ be absolutely continuous and have a derivative $y_0'(t)$ which is essentially of bounded variation on $0 \leq t \leq 1$, and let $F(x)$ be a Wiener summable functional over $C$. Let $F(x)$ have a first variation $\delta F = \delta F(x | y_0)$ for all $x \in C$ such that

$$\sup_{|h| \leq \eta} |\delta F(x + hy_0 | y_0)|$$

is Wiener summable in $x$ on $C$ for some $\eta>0$. Then it follows that

$$\int_C \delta F(x | y_0) dwx = 2 \int_C F(x) \left[ \int_0^1 y_0'(t) dx(t) \right] dwx.$$

As a corollary to Theorem II we obtain a formula for "integration by parts in function space." We replace $P(x)$ by $F(x)G(x)$.

**Corollary.** Let $y_0(t)$ be absolutely continuous and have a derivative $y_0'(t)$ which is essentially of bounded variation on $0 \leq t \leq 1$, and let $F(x)$ and $G(x)$ be Wiener measurable functionals on $C$ such that $F(x)G(x)$ is Wiener summable on $C$. Let $F$ and $G$ have first variations $\delta F$ and $\delta G$ such that

$$G(x) \sup_{|h| \leq \eta} |\delta F(x + hy_0 | y_0)| \quad \text{and} \quad F(x) \sup_{|h| \leq \eta} |\delta G(x + hy_0 | y_0)|$$

are Wiener summable in $x$ on $C$ for some $\eta>0$. Then it follows that

$$\int_C F(x)\delta G(x | y_0) dwx$$

is

$$= \int_C G(x) \left[ 2F(x) \int_0^1 y_0'(t) dx(t) - \delta F(x | y_0) \right] dwx.$$

Theorem III. Let $F(x)$ be a Wiener summable functional such that $F(x) \max_{0 \leq t \leq 1} |x(t)|$ is also Wiener summable, and such that the first variation

$$\delta F = \delta F(x \mid y) = \frac{d}{dh} F(x + hy) \bigg|_{h=0}$$

exists for all $x$ and $y$ in $C$ and is expressible in the form

$$\delta F(x \mid y) = \int_0^1 F'(x \mid t) y(t) dt,$$

where $F'(x \mid t)$ is measurable in the product space of $x$ and $t$ as well as summable in $t$ for each $x$. (It is clear that $F'(x \mid t)$ is the Volterra derivative of $F(x)$ at each point $(x, t)$ of the product space for which $F'(x \mid t)$ is continuous in $(x, t)$.) Assume also that for each $y(t) \in C$ there exists a corresponding number $\eta = \eta(y) > 0$ such that

$$\sup_{|h| \leq \eta, 0 \leq t \leq 1} |F'(x + hy \mid t)|$$

is Wiener summable in $x$ on $C$. Then it follows that $\int_C F(x) x(t) dw x$ has an absolutely continuous derivative with respect to $t$ for $0 \leq t \leq 1$, and this derivative vanishes at $t = 1$. Moreover

$$\int_0^1 F'(x \mid t) dt = -2 \frac{d^2}{dt^2} \int_C F(x) x(t) dw x$$

for almost all $t$ on $0 \leq t \leq 1$, and, in particular, for each $t$ for which the left member is continuous. Specifically, (19) holds for each $t$ for which $F'(x \mid t)$ is continuous in $t$ for almost all $x$ in $C$.

We shall prove that this theorem holds even when we weaken the hypotheses (16), (17), (18) by assuming that they hold not for all $y$ in $C$, but only for a sequence of values of $y$, namely $y = y_n \ (n = 1, 2, \cdots)$, where each $y_n(t)$ has an absolutely continuous derivative $y_n'(t)$ and satisfies $y_n(0) = y_n'(1) = 0$, and where the set of second derivatives $\{y_n''(t)\}$ is closed in $L_2$ on $0 \leq t \leq 1$. Then if $||y|| = \max_{0 \leq t \leq 1} |y(t)|$, we have by (17) for each $n = 1, 2, \cdots$ the inequality

$$\sup_{|h| \leq \eta_n, 0 \leq t \leq 1} |\delta F(x + hy_n \mid t)| \leq \eta_n \cdot \sup_{|h| \leq \eta_n, 0 \leq t \leq 1} |F'(x + hy_n \mid t)|,$$

where $\eta_n$ denotes $\eta(y_n)$. Thus for each $n$, (18) and (20) imply that
the hypotheses of Theorem II hold with \( y_n \) replacing \( y_0 \), and we have from \((15)\),

\[
\int_C \delta F(x \mid y_n) \, dx = 2 \int_C \left[ \int_0^1 y_n'(t) \, dx(t) \right] \, dx.
\]

Integrating by parts in the right member, remembering that \( x(0) = y_n'(1) = 0 \), and using \((17)\) in the left member, we obtain

\[
\int_C \left[ \int_0^1 F'(x \mid t) \, y_n(t) \, dt \right] \, dx = -2 \int_C \left[ \int_0^1 x(t) \, y_n''(t) \, dt \right] \, dx.
\]

Since by hypothesis \( F(x) \cdot \|x\| \) is summable, we may apply the Fubini theorem to the right member, and since \((18)\) is summable we may apply it to the left member. Thus

\[(21) \quad \int_0^1 y_n(t) \Psi(t) \, dt = -2 \int_0^1 y_n''(t) \, dt \int_C F(x) x(t) \, dx,
\]

where

\[(22) \quad \Psi(t) = \int_C F'(x \mid t) \, dx.
\]

We next integrate the left member of \((21)\) by parts twice, and to simplify the notation we introduce the function

\[
\phi(t) = \int_0^t \int_1^u \int_0^1 F'(x \mid s) \, ds \, dx \, du
\]

which obviously satisfies the conditions

\[(23) \quad \phi(t) \text{ and } \phi'(t) \text{ absolutely continuous on } 0 \leq t \leq 1,
\]

\[
\phi''(t) = \Psi(t) \text{ almost everywhere on } 0 \leq t \leq 1,
\]

\[
\phi(0) = \phi'(1) = 0.
\]

Thus we obtain from \((21)\) by two integrations by parts, using \( y_n(0) = y_n'(1) = 0 \) and \((23)\),

\[(24) \quad \int_0^1 y_n''(t) \left[ \phi(t) + 2 \int_C F(x) x(t) \, dx \right] \, dt = 0.
\]

But the \( y_n''(t) \) are closed (and hence complete) in \( L_2(0, 1) \), and therefore \((24)\) implies that for almost all \( t \) on \( 0 \leq t \leq 1 \)
\[ (25) \quad \phi(t) = -2 \int_{C}^{W} F(x)x(t) d\mu x. \]

Actually, (25) is true for all \( t \) on the unit interval, since both sides are continuous. The continuity of the right member follows from the continuity of \( x(t) \) and the summability of \( F(x)||x|| \). Differentiating (25), we obtain for all \( t \) on \( 0 \leq t \leq 1 \),

\[ \phi'(t) = -2 \frac{d}{dt} \int_{C}^{W} F(x)x(t) d\mu x. \]

From this and (23) it is clear that \( \int_{C}^{W} F(x)x(t) d\mu x \) has an absolutely continuous derivative on \( 0 \leq t \leq 1 \) which vanishes at \( t = 1 \). Another differentiation gives (19) for almost all \( t \), and in particular whenever the left member is continuous. This must occur in view of (18) whenever \( F'(x|t) \) is continuous in \( t \) for almost all \( x \), and hence the theorem is established.

**Corollary.** Theorem III holds when hypotheses (16), (17), (18) are assumed to hold only for a sequence of \( y(t), \{ y_n(t) \} \), such that each \( y_n \) has an absolutely continuous derivative and \( y_n(0) = y_n'(1) = 0 \) and the second derivatives \( \{ y_n''(t) \} \) are closed in \( L_2(0, 1) \).

**Example.** We conclude this paper by giving an example to show how Theorem II can be used to evaluate new Wiener integrals.

As our example, let

\[ F(x) = \log \left[ \int_{0}^{1} \alpha(t) \exp \left\{ \frac{x(t)}{\beta(t)} \right\} dt \right], \]

where the integrand is understood to vanish when \( \alpha(t) \) vanishes whether the exponential exists or not, and where \( \alpha(t) \) and \( \beta(t) \) satisfy the following conditions. We assume \( \alpha(t) \in L_1, \alpha(t) \geq 0 \) on \( 0 \leq t \leq 1 \), \( \alpha(t) > 0 \) on a set of positive measure; \( \beta(t) \in C \) and is absolutely continuous with a derivative essentially of bounded variation; finally, we assume

\[ \int_{0}^{1} \alpha(t) \exp \left\{ \frac{t}{4[\beta(t)]^2} \right\} dt < \infty; \]

where the integrand is (as above) interpreted to vanish when \( \alpha(t) \) vanishes.

We first note that \( F(x) \in L_\nu(C) \) for all positive \( p \). For if \( r > 0 \),

\[ |\log r|^p < \tilde{p} \max (r, r^{-1}) \]
and hence
\[
| F(x) |^p < p^p \max \left[ \int_0^1 \alpha(t) \exp \left( \frac{x(t)}{\beta(t)} \right) dt, \right.
\]
\[
\left. \left\{ \int_0^1 \alpha(t) \exp \left[ - \frac{x(t)}{\beta(t)} \right] dt \right\}^{-1} \right].
\]
Moreover by the Schwartz inequality
\[
\int_0^1 \alpha(t) \exp \left\{ - \frac{x(t)}{\beta(t)} \right\} dt \cdot \int_0^1 \alpha(t) \exp \left[ \frac{x(t)}{\beta(t)} \right] dt \geq \left[ \int_0^1 \alpha(t) dt \right]^2,
\]
so that
\[
| F(x) |^p < K \int_0^1 \alpha(t) \exp \left( \frac{x(t)}{\beta(t)} \right) dt
\]
where
\[
K = p^p \max \left[ 1, \left( \int_0^1 \alpha(t) dt \right)^{-2} \right].
\]
Thus
\[
\int_c^w | F(x) |^p dx < K \int_0^1 \alpha(t) dt \int_c^w \exp \left( \frac{x(t)}{\beta(t)} \right) dx
\]
\[
= \frac{K}{\pi^{1/2}} \int_0^1 \alpha(t) dt \int_{-\infty}^\infty \exp \left[ \frac{s}{\beta(t)} - \frac{s^2}{t} \right] ds
\]
\[
< 2K \int_0^1 \alpha(t) dt \int_{-\infty}^\infty \cosh \left( \frac{u t^{1/2}}{\beta(t)} \right) e^{-u^2} du
\]
\[
= 2K \int_0^1 \alpha(t) \exp \left\{ \frac{t}{4[\beta(t)]^2} \right\} dt
\]
\[
< \infty.
\]
Now let \( \mathcal{N} \) be the null set where \( F(x) \) fails to exist, and let us define \( F(x) \) to be zero on \( \mathcal{N} \). We then have
\[
F(x + h \beta) = \begin{cases} 
F(x) + h & \text{when } x \in C - \mathcal{N}, \\
0 & \text{when } x \in \mathcal{N},
\end{cases}
\]
\[
\delta F(x \mid \beta) = \begin{cases} 
1 & \text{when } x \in C - \mathcal{N}, \\
0 & \text{when } x \in \mathcal{N}.
\end{cases}
\]
Thus the hypotheses of Theorem II (with \( \gamma_0 = \beta \)) are satisfied and we obtain from (15)
\[
\int_C \log \left[ \int_0^1 a(t) \exp \left( \frac{x(t)}{\beta(t)} \right) dt \right] \left[ \int_0^1 \beta'(t) dx(t) \right] dW x = \frac{1}{2}.
\]
In particular, if \( a(t) = 0 \) when \( t < s \) and \( \beta(t) = s^{-1} \min(s, t) \) for some fixed \( s \) on \( 0 < s < 1 \), we obtain formula (2) given in the introduction.

As another special case, take \( \beta(t) = t/2 \) and \( a(t) = \phi(t) \exp(-t^{-1}) \), where \( \phi(t) \) is non-negative and summable and not equivalent to zero on \( 0 \leq t \leq 1 \). Clearly the required conditions on \( a \) and \( \beta \) are satisfied, and we have
\[
\int_C x(1) \left[ \log \int_0^1 \phi(t) \exp \left( \frac{2x(t) - 1}{t} \right) dt \right] dW x = 1.
\]
Other interesting formulas can be obtained by using the formula for “integrating by parts in function space.”

**Bibliography**


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