MOTIONS IN LINEARLY CONNECTED TWO-DIMENSIONAL SPACES

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1. **Introduction.** A linearly connected space L_n is characterized by its coefficients of (asymmetric) connection L^i_{jk} which transform according to the law [2, p. 3][1]

$$L^i_{jk}(x) \frac{\partial \bar{x}^a}{\partial x^i} = \bar{L}^a_{bc}(\bar{x}) \frac{\partial \bar{x}^b}{\partial x^j} \frac{\partial \bar{x}^c}{\partial x^k} + \frac{\partial^2 \bar{x}^a}{\partial x^j \partial x^k}. \tag{1.1}$$

An L_n will admit an infinitesimal motion defined by the infinitesimal transformation

$$\bar{x}^i = x^i + \xi^i(x)\delta t \tag{1.2}$$

if

$$\frac{\partial^2 \xi^i}{\partial x^j \partial x^k} + \frac{\partial \Gamma^i_{jk}}{\partial x^h} \xi^h + \Gamma^i_{hk} \frac{\partial \xi^h}{\partial x^j} + \Gamma^i_{jh} \frac{\partial \xi^h}{\partial x^k} - \Gamma^h_{jk} \frac{\partial \xi^i}{\partial x^h} = 0, \tag{1.3a}$$

$$\frac{\partial \Omega^i_{jk}}{\partial x^h} \xi^h + \Omega^i_{hk} \frac{\partial \xi^h}{\partial x^j} + \Omega^i_{jh} \frac{\partial \xi^h}{\partial x^k} - \Omega^h_{jk} \frac{\partial \xi^i}{\partial x^h} = 0, \tag{1.3b}$$

where

$$L^i_{jk} = \Gamma^i_{jk} + \Omega^i_{jk}, \tag{1.4}$$

and Γ^i_{jk}, Ω^i_{jk} are the symmetric and skew-symmetric parts respectively of L^i_{jk} [1, p. 231].

In case $\Omega^i_{jk}=0$, the L_n reduces to a symmetrically connected space A_n in which (1.3a) defines the infinitesimal affine collineations [2, p. 125], so that motions in an L_n can be considered as generalizations of such collineations of an A_n [3].

In a previous paper [3] we obtained all (symmetric) two-dimensional spaces A_2 admitting real continuous groups of affine collineations, these being obtained by solving (1.3a) for the Γ^i_{jk}, the ξ^i being known and obtained from Lie's classification [4, pp. 71–73, 379–380] giving all real continuous groups G_r in two variables. In §2 we carry through a like procedure to obtain all linearly connected L_2 admitting complete groups of motions. This will involve the solution of the

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

combined system (1.3a–b) for the Γ^i_{jk} and Ω^i_{jk}. The results of [3] will of course eliminate much of the analysis which would otherwise be necessary. We shall exclude all solutions of the system (1.3) which lead to A_2 spaces ($\Omega^i_{jk}=0$).

In §3 we consider the special case of spaces of absolute parallelism T_n, and determine all T_2 admitting motion groups. The conditions on an L_n to reduce to a T_n are [1, p. 234]

$$L^{i}_{jkm} \equiv \frac{\partial L^{i}_{jm}}{\partial x^k} - \frac{\partial L^{i}_{jk}}{\partial x^m} + L^{h}_{jm}L^{i}_{hk} - L^{h}_{jk}L^{i}_{hm} = 0, \tag{1.5}$$

that is, the L_n is of zero curvature. In this case there exists an ennuple of (absolutely) parallel vector fields $h^i_{\alpha|}$, that is,

$$h^{i}_{\alpha|j} \equiv \frac{\partial h^{i}_{\alpha|}}{\partial x^j} + h^{m}_{\alpha|}L^{i}_{mj} = 0 \qquad (\alpha = 1, \cdots, n). \tag{1.6}$$

These equations (1.6) are solved for the $h^i_{\alpha|}$ using for the L^i_{jk} the values found from the solutions of (1.3). (The integrability conditions of (1.6) are (1.5).)

The problem of motions in a T_n has been treated by Robertson [5] who obtained the conditions for infinitesimal motions of a T_n in the form

$$\frac{\partial h^{i}_{\alpha|}}{\partial x^j}\xi^j - h^{j}_{\alpha|}\frac{\partial \xi^i}{\partial x^j} = \epsilon^{\beta}_{\alpha}h^{i}_{\beta|}, \tag{1.7}$$

where ϵ^β_α are constants. (See also [6].)

It is shown in §2 that there are 8 possible types of L_2 admitting groups of motions, the maximum number of parameters being four. In §3 it is shown there are 7 types of motions for a T_2.

2. **Motions in an L_2.** We proceed to the solution of (1.3) for the case $n=2$. It is easily shown that (1.3b) reduces to

$$\frac{\partial A}{\partial x}\xi^1 + \frac{\partial A}{\partial y}\xi^2 + A\frac{\partial \xi^2}{\partial y} - B\frac{\partial \xi^1}{\partial y} = 0, \tag{2.1a}$$

$$\frac{\partial B}{\partial x}\xi^1 + \frac{\partial B}{\partial y}\xi^2 + B\frac{\partial \xi^1}{\partial x} - A\frac{\partial \xi^2}{\partial x} = 0, \tag{2.1b}$$

where

$$A = \overset{1}{\Omega}_{12}, \qquad B = \overset{2}{\Omega}_{12} \qquad (x, y = \overset{1}{x}, \overset{2}{x}). \tag{2.2}$$

Corresponding to each G_r of Lie's classification we solve (2.1) for A and B, excluding all solutions for which $A=B=0$. (The solutions of (1.3a) for the Γ^i_{jk} have been obtained in [3].) For each of the nonexcluded solutions the G_r is tested for completeness as explained in [3].

As an illustration we consider the $G_3[p, q, xq]$ where $p=\partial f/\partial x$, $q=\partial f/\partial y$. Using $X_1f=p$, $X_2f=q$, $X_3f=xq$, with $\xi^i_{1|}=\delta^i_1$, $\xi^i_{2|}=\delta^i_2$, $\xi^1_{3|}=0$, $\xi^2_{3|}=x$ in (2.1) we obtain $A=0$, $B=B_0$ an arbitrary constant.

To obtain the complete group, put $A=0$, $B=B_0$ in (2.1), and solve for ξ^i. This gives $\xi^1=k$, $\xi^2=\xi^2(x, y)$, with k an arbitrary constant and ξ^2 an arbitrary function. Now as determined in [3] the Γ's obtained as solutions of (1.3a) corresponding to the above G_3 will admit as their complete group the $G_4[p, q, xq, yq]$. It is thus seen that by taking $k=0$, $\xi^2=y$, we obtain the fourth generator yq of the G_4, and hence this G_4 will be the complete group of motions of the L_2 given by

$$\overset{1}{L}_{11} = 2a, \qquad \overset{1}{L}_{12} = \overset{1}{L}_{21} = \overset{1}{L}_{22} = \overset{2}{L}_{11} = \overset{2}{L}_{22} = 0,$$

$$\overset{2}{L}_{12} = a + B_0, \quad \overset{2}{L}_{21} = a - B_0 \qquad (B_0 \neq 0).$$

The Γ's used to give these L's are obtained from group [4.1] of [3].

By proceeding as above with each G_r and making use of the results of [3] we find eight possible types of L_2 spaces admitting real groups of motions. These with their respective groups are given below.

Asymmetrically connected space L_2 and their complete groups of motions

[A1.1] $[p]$

$$\overset{i}{L}_{jk} = \overset{i}{L}_{jk}(y) \qquad \text{(arbitrary functions of } y).$$

[A2.1] $[p, q]$

$$\overset{i}{L}_{jk} = \text{arbitrary constants.}$$

[A2.2] $[p, xp + yq]$

$$\overset{i}{L}_{jk} = (1/y)\overset{i}{A}_{jk} \qquad (\overset{i}{A}_{jk} \text{ arbitrary constants}).$$

[A3.1] $[p, 2xp + yq, x^2p + xyq]$

$$\overset{1}{L}_{11} = \frac{2b}{y^2}, \quad \overset{1}{L}_{12} = -\frac{1}{y}, \quad \overset{1}{L}_{21} = -\frac{1}{y}, \quad \overset{1}{L}_{22} = 0,$$

$$\overset{2}{L}_{11} = \frac{a}{y^3}, \quad \overset{2}{L}_{12} = \frac{b+c}{y^2}, \quad \overset{2}{L}_{21} = \frac{b-c}{y^2}, \quad \overset{2}{L}_{22} = -\frac{2}{y} \quad (c \neq 0).$$

[A3.2] $[q,\ xq,\ yq]$

$$\overset{1}{L}_{11} = 2f(x),\quad \overset{1}{L}_{12} = \overset{1}{L}_{21} = \overset{1}{L}_{22} = \overset{2}{L}_{11} = \overset{2}{L}_{22} = 0,$$

$$\overset{2}{L}_{12} = f(x) + B(x),\quad \overset{2}{L}_{21} = f(x) - B(x) \qquad (B \neq 0).$$

[A4.1] $[p,\ q,\ xq,\ yq]$

$$\overset{1}{L}_{11} = 2a,\quad \overset{1}{L}_{12} = \overset{1}{L}_{21} = \overset{1}{L}_{22} = \overset{2}{L}_{11} = \overset{2}{L}_{22} = 0,$$

$$\overset{2}{L}_{12} = a + b,\quad \overset{2}{L}_{21} = a - b \qquad (b \neq 0).$$

[A4.2] $[p,\ q,\ yq,\ e^{x}q]$

$$\overset{1}{L}_{11} = 1/3 + 2a,\quad \overset{1}{L}_{12} = \overset{1}{L}_{21} = \overset{1}{L}_{22} = \overset{2}{L}_{11} = \overset{2}{L}_{22} = 0,$$

$$\overset{2}{L}_{12} = a + c - 1/3,\quad \overset{2}{L}_{21} = a - c - 1/3 \qquad (c \neq 0).$$

[A4.3] $[p,\ yq,\ (e^{ax}\cos x)q,\ \ (e^{ax}\sin x)q]$

$$\overset{1}{L}_{11} = 2a/3 + 2b,\quad \overset{1}{L}_{12} = \overset{1}{L}_{21} = \overset{1}{L}_{22} = \overset{2}{L}_{22} = 0,$$

$$\overset{2}{L}_{11} = (1 + a^{2})y,\quad \overset{2}{L}_{12} = -\ 2a/3 + b + c,$$

$$\overset{2}{L}_{21} = -\ 2a/3 + b - c \qquad (c \neq 0).$$

In the above a, b, c are arbitrary constants, $f(x)$, $B(x)$ are arbitrary functions subject to indicated restrictions.

3. **Motions in spaces T_2 of absolute parallelism.** For each of the 8 types of L_2 obtained in the previous section, we impose conditions (1.5), and then solve (1.6) for the $\overset{i}{h}_{\alpha|}$.

The case [A3.2] will be used as an illustration. Conditions (1.5) reduce to $(f+B)'-(f+B)^2=0$, so that $f(x)+B(x)=-1/(x+c)$ with c an arbitrary constant. Equations (1.6) are

$$\text{(3.1a)}\quad \frac{\partial h^1}{\partial x} + 2h^1 f(x) = 0, \qquad \text{(3.1c)}\quad \frac{\partial h^2}{\partial x} + h^2(f(x) - B(x)) = 0,$$

$$\text{(3.1b)}\quad \frac{\partial h^1}{\partial y} = 0, \qquad \text{(3.1d)}\quad \frac{\partial h^2}{\partial y} - \frac{h^1}{x + c} = 0,$$

which can be solved to give $h^1=ag(x)$, $h^2=(ay+b)g(x)/(x+c)$, where $g(x)=\exp\{-2\int f(x)dx\}$.

The complete solution is then

$$\text{(3.2)}\qquad \overset{1}{h}_{\alpha|} = a_\alpha g(x),\qquad \overset{2}{h}_{\alpha|} = \frac{a_\alpha y + b_\alpha}{x + c}\, g(x) \qquad (\alpha = 1,\ 2),$$

with a_α, b_α, c arbitrary constants such that $a_1b_2-a_2b_1\neq 0$. (The special case $\overset{2}{L}_{12}=f+B=0$ gives the solution $\overset{1}{h}_{\alpha|}=a_\alpha g(x)$, $\overset{2}{h}_{\alpha|}=b_\alpha g(x)$.)

It is of interest to verify that the T_2 will admit the finite group generated by the infinitesimal group. To show this we first obtain the finite equations of the group, these being

$$\bar{x} = x, \qquad \bar{y} = \lambda x + \mu y + \nu \tag{3.3}$$

with the 3 parameters λ, μ, ν.

Now we must have [5]

$$\overset{i}{h}_{\alpha|}(\bar{x}, \bar{y}) = \overset{\beta}{A}_{\alpha}\overset{j}{h}_{\beta|}(x, y)\frac{\partial \bar{x}^i}{\partial x^j}, \tag{3.4}$$

where the A^{β}_{α} must be functions $A^{\beta}_{\alpha}(\lambda, \mu, \nu)$ of the parameters only. From (3.2), (3.3), and (3.4) we obtain the relations

$$\overset{\beta}{A}_{\alpha}a_{\beta} = a_{\alpha}, \qquad \overset{\beta}{A}_{\alpha}b_{\beta} = c_{\alpha}, \tag{3.5}$$

where $c_\alpha = [a_\alpha(\nu - \lambda c) + b_\alpha]/\mu$. It follows that the A^{β}_{α} exist and are of the required form.

The results from the solutions of (1.6) are given below.

Spaces T_2 of absolute parallelism admitting complete groups of motions

[T1.1] [*p*]

$$\text{(a)}\quad \overset{1}{h}_{\alpha|} = \overset{j}{a}_{\alpha}\exp\left\{\int(\lambda_j \overset{1}{L}_{2i} - \overset{1}{L}_{1i})dx^i\right\},$$

$$\overset{2}{h}_{\alpha|} = -\sum_j \overset{j}{a}_{\alpha}\lambda_j \exp\left\{\int(\lambda_j \overset{1}{L}_{2i} - \overset{1}{L}_{1i})dx^i\right\};$$

$$\text{(b)}\quad \overset{1}{h}_{\alpha|} = \exp\left\{\int(\lambda \overset{1}{L}_{2i} - \overset{1}{L}_{1i})dx^i\right\}[a_\alpha \cos\rho + b_\alpha \sin\rho],$$

$$\overset{2}{h}_{\alpha|} = -\exp\left\{\int(\lambda \overset{1}{L}_{2i} - \overset{1}{L}_{1i})dx^i\right\}[a_\alpha(\lambda\cos\rho - \mu\sin\rho) + b_\alpha(\lambda\sin\rho + \mu\cos\rho)];$$

$$\text{(c)}\quad \overset{1}{h}_{\alpha|} = \exp\left\{\int(\lambda \overset{1}{L}_{2i} - \overset{1}{L}_{1i})dx^i\right\}\left[a_\alpha\int\frac{\overset{1}{L}_{2i}dx^i}{\overset{1}{L}_{21}} + b_\alpha\right],$$

$$\overset{2}{h}_{\alpha|} = -\exp\left\{\int(\lambda \overset{1}{L}_{2i} - \overset{1}{L}_{1i})dx^i\right\} \cdot\left[\frac{a_\alpha}{\overset{1}{L}_{21}} + \lambda\left(a_\alpha\int\frac{\overset{1}{L}_{2i}dx^i}{\overset{1}{L}_{21}} + b_\alpha\right)\right].$$

In (a), $\lambda_1 \neq \lambda_2$ are the real roots of $f(\lambda) \equiv \lambda^2 \overset{1}{L}_{21} + \lambda(\overset{2}{L}_{21} - \overset{1}{L}_{11}) - \overset{2}{L}_{11} = 0$;

in (b), $\lambda \pm i\mu$ are its roots; and in (c), $f(\lambda)$ is assumed to have equal roots, λ, λ. In (b), $\rho = \int \mu L^1_{2i} dx^i$. The $L^i_{jk}(y)$ are otherwise arbitrary subject to (1.5), which have as a consequence that $L_{11} + L_{21}$ and $L_{11}L_{21} - L_{21}L_{11}$ must be constant. The a_α, b_α, a^j_α are arbitrary constants such that $h^i_{\alpha|}$ are linearly independent. For special values of the L^i_{jk}, for example, $L^1_{21} = 0$, the solutions for the $h^i_{\alpha|}$ can be easily obtained.

[T2.1] $[p, q]$

The $h^i_{\alpha|}$ are obtained as in the 3 cases of [T1.1] except the L^i_{jk} are now considered constants.

[T2.2] $[p, xp + yq]$

$$\overset{1}{h}_{\alpha|} = y^m(a_\alpha u + b_\alpha), \quad \overset{2}{h}_{\alpha|} = y^m\ (c_\alpha v + d_\alpha),$$

with $c_\alpha A^2_{11} + a_\alpha A^1_{11} = 0$, $d_\alpha A^1_{21} + b_\alpha A^1_{11} = 0$, $m = -(1 + A^1_{12} + A^2_{22})/2$, and $u = xA^1_{21} + yA^1_{22}$, $v = xA^2_{11} + yA^2_{12}$. The constants A^i_{jk} are arbitrary subject to (1.5). As in [T1.1] the solutions for $h^i_{\alpha|}$ corresponding to special values of the A^i_{jk} can be obtained without difficulty.

[T3.1] $[p, 2xp + yq, x^2p + xyq]$

$$\overset{1}{h}_{\alpha|} = y(a_\alpha x + b_\alpha), \quad \overset{2}{h}_{\alpha|} = (y^2 + 2bx)a_\alpha + 2bb_\alpha.$$

The a, b, c of [A3.1] must satisfy $c = 3b$, $a = 4b^2$, $b \neq 0$.

[T3.2] $[q, xq, yq]$

$$\overset{1}{h}_{\alpha|} = a_\alpha g(x), \quad \overset{2}{h}_{\alpha|} = (a_\alpha y + b_\alpha)g(x)/(x + c).$$

Here the L^i_{jk} of [A3.2] satisfy $g(x) = \exp\{-\int 2f(x)dx\}$, and $f(x) + B(x) = -1/(x + c)$.

[T4.1] $[p, q, xq, yq]$

$$\overset{1}{h}_{\alpha|} = a_\alpha e^{-2ax}, \quad \overset{2}{h}_{\alpha|} = b_\alpha e^{-2ax},$$

with $a + b = 0$, $(b \neq 0)$ in [A4.1].

[T4.2] $[p, q, yq, e^x q]$

$$\overset{1}{h}_{\alpha|} = a_\alpha e^{(2c-1)x}, \quad \overset{2}{h}_{\alpha|} = b_\alpha e^{2cx} \qquad (a + c = 1/3 \text{ in [A4.2]}),$$

$$\overset{1}{h}_{\alpha|} = a_\alpha e^{(2c+1)x}, \quad \overset{2}{h}_{\alpha|} = (a_\alpha y + b_\alpha)e^{(2c+1)x}$$

$$(a + c = -2/3 \text{ in [A4.2]}).$$

Bibliography

1. L. P. Eisenhart, *Continuous groups of transformations*, Princeton University Press, 1933.

2. ———, *Non-Riemannian geometry,* Amer. Math. Soc. Colloquium Publications, vol. 8, 1927.

3. J. Levine, *Classification of collineations in projectively and affinely connected spaces of two dimensions,* Ann. of Math. vol. 52 (1950) pp. 465–477.

4. S. Lie and F. Engel, *Theorie der Transformationsgruppen,* vol. 3, Leipzig, Teubner, 1893.

5. H. P. Robertson, *Groups of motions in spaces admitting absolute parallelism,* Ann. of Math. vol. 33 (1932) pp. 496–520.

6. K. Yano, *Groups of transformations in generalized spaces,* Tokyo, Akademeia Press, 1949.

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