A SOLUTION OF THE "PLANK PROBLEM"

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The "plank problem" of Tarski[1] is the following: Let L be a convex domain of minimal width l; is it then true, when L is entirely covered by p parallel strips with the widths $h_1, \cdots, h_p$, that $h_1 + \cdots + h_p \geqq l$?

We shall here give an affirmative answer to this question.[2] We shall prove the following general theorem:

THEOREM. *If L is a convex body of minimal width l in the n-dimensional Euclidean space, and L is contained in the union of p parallel strips of widths $h_1, \cdots, h_p$, then $h_1 + \cdots + h_p \geqq l$.*

By a parallel strip of width h we mean the part of the space lying between two parallel $(n-1)$-dimensional hyperplanes whose distance is h.

Let M be a domain and $\bar{v}$ a vector in the n-dimensional space; $M + \bar{v}$ shall denote the domain into which M is translated by the vector.

LEMMA 1. *Let M be a convex domain of minimal width m, and $\bar{v}$ a vector of length $|\bar{v}| = h/2 < m/2$; then the intersection $\bigcap(M \mp \bar{v}) = (M - \bar{v}) \cap (M + \bar{v})$, which is convex, is not empty, and its minimal width is not less than $m - h$.*

PROOF. M has at least one chord in the direction $\bar{v}$ with parallel supporting hyperplanes at its ends. It is obvious that the length k of this chord is not less than m. A length $k-h$ of this chord is contained in the intersection. Let A on the boundary of $M+\bar{v}$ and B on the boundary of $M-\bar{v}$ be the ends of this length $k-h$. If A is taken as center of a similitude of the ratio $(k-h)/k < 1$, then $M+\bar{v}$ is carried into a part of itself, and the minimal width of this part is $m(k-h)/k \geqq m-h$. Since this new domain is also obtained from $M-\bar{v}$ through a similitude of the same ratio and with its center in B,

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[1] The problem was first posed by Tarski in *Uwagi o stopniu równowazności wielokatów*, Parametr vol. 2 (1932). In the same paper it was stated that the solution of the problem is positive for a circle (and more generally for any two-dimensional figure in which a circle of an equal width can be inscribed). The solution was based upon an idea of Mr. H. Moese, developed in an article which also appeared in Parametr vol. 2 (1932).

[2] The following proof is a simplification and generalization of a proof given by the author in *On covering by parallel-strips*, Matematisk Tidsskrift B (1950) pp. 49–53.

it is contained in $M-\bar{v}$, and hence in the intersection. This proves the lemma.

Now let us consider parallel strips. They shall be directed, such that one of their enclosing hyperplanes will be denoted positive, and the other negative. To each directed parallel strip corresponds a vector $\bar{u}$ and a constant c, such that the equation of the positive hyperplane is $\bar{r}\cdot\bar{u}+c=\bar{u}^2$ (where $\bar{r}$ is the variable point) and the equation of the negative hyperplane is $\bar{r}\cdot\bar{u}+c=-\bar{u}^2$. The vector $\bar{u}$ is perpendicular to the strip, it is directed from the negative towards the positive hyperplane, and its length is the semiwidth of the strip. We have thus obtained a one-to-one correspondence between the directed parallel strips of the space and the sets $(\bar{u}, c)$ (where $|\bar{u}|>0$). If the direction of the strip is changed, then $(\bar{u}, c)$ is replaced by $(-\bar{u}, -c)$.

Now let us consider p parallel strips $(\bar{u}_1, c_1), \cdots, (\bar{u}_p, c_p)$. The part of the space not contained in the union of the strips consists of polyhedrons (finite or infinite); let us denote them by $P_{\epsilon_1\cdots\epsilon_p}$ where ϵ_j is $+1$ or -1 for the polyhedron on the positive or the negative side of the strip $(\bar{u}_j, c_j)$, respectively. The finite sequence $\epsilon_1, \cdots, \epsilon_p$ will be denoted by ϵ, and this ϵ shall run through all the 2^p sequences formed by the numbers $+1$ and -1 (even if some of them do not correspond to existing polyhedrons P_ϵ). When we in the following apply linear operations to such sequences ϵ, the sense shall be the same as if they were p-dimensional vectors.

Let δ be a finite sequence of p numbers $\delta_1, \cdots, \delta_p$; let δc stand for the "inner product" $\delta_1c_1+\cdots+\delta_pc_p$, and $\delta\bar{u}$ for $\delta_1\bar{u}_1+\cdots+\delta_p\bar{u}_p$. Let H_δ denote the set of points $\bar{r}$ which satisfy the inequality $\bar{r}\cdot(\delta\bar{u})+\delta c\geqq(\delta\bar{u})^2$. When $|\delta\bar{u}|>0$, then H_δ is the half-space on the positive side of the strip $(\delta\bar{u}, \delta c)$.

Lemma 2. *The union $\bigcup_\epsilon(P_\epsilon-\epsilon\bar{u})$ is the whole space.*

Proof. For a given $\epsilon=(\epsilon_1, \cdots, \epsilon_p)$, let $\delta^{(j)}$ be the sequence $(0, \cdots, \epsilon_j, \cdots, 0)$ with zeros everywhere except in the jth place. Then

$$P_\epsilon = \bigcap_j H_{\delta^{(j)}}.$$

Let us define Q_ϵ by

$$Q_\epsilon = \bigcap_{\epsilon'} H_{(\epsilon-\epsilon')/2}.$$

Since each $\delta^{(j)}$ can be written in the form $(\epsilon-\epsilon')/2$, we have $Q_\epsilon\subset P_\epsilon$, and we shall now prove that $\bigcup_\epsilon(Q_\epsilon-\epsilon\bar{u})$ is the whole space.

A point $\bar{r}$ lies in $H_{(\epsilon-\epsilon')/2}-\epsilon\bar{u}$ when, and only when, $\bar{r}+\epsilon\bar{u}$ satisfies the definition of $H_{(\epsilon-\epsilon')/2}$, that is, when

$$(\bar{r}+\epsilon\bar{u})\cdot\left(\frac{\epsilon-\epsilon'}{2}\,\bar{u}\right)+\frac{\epsilon-\epsilon'}{2}\,c \geqq \left(\frac{\epsilon-\epsilon'}{2}\,\bar{u}\right)^2$$

or

$$2\bar{r}\cdot(\epsilon\bar{u})+2\epsilon c+(\epsilon\bar{u})^2 \geqq 2\bar{r}\cdot(\epsilon'\bar{u})+2\epsilon' c+(\epsilon'\bar{u})^2;$$

and $\bar{r}$ will lie in $Q_\epsilon-\epsilon\bar{u}$ when this inequality is satisfied for all ϵ'. Hence, every point $\bar{r}$ will lie in a $Q_\epsilon-\epsilon\bar{u}$, namely for that (or one of the) ϵ for which the linear expression $2\bar{r}\cdot(\epsilon\bar{u})+2\epsilon c+(\epsilon\bar{u})^2$ is maximal. This proves the lemma.

Now we can prove our theorem: Given p parallel strips $(\bar{u}_1, c_1)$, $\cdots$, $(\bar{u}_p, c_p)$, with widths respectively $h_1=2|\bar{u}_1|, \cdots, h_p=2|\bar{u}_p|$. Suppose that $h_1+\cdots+h_p<l$, where l is the minimal width of the convex body L; we shall prove that L is not entirely contained in the union of the strips. The part of L outside the union of the strips consists of the domains $L\cap P_\epsilon$. We give the domain $L\cap P_\epsilon$ a translation $-\epsilon\bar{u}$, and then the total part of the space filled by them is

$$\bigcup_\epsilon\,[(L-\epsilon\bar{u})\cap(P_\epsilon-\epsilon\bar{u})] \supset \bigcup_\epsilon\,[\{\bigcap_{\epsilon'}(L-\epsilon'\bar{u})\}\cap(P_\epsilon-\epsilon\bar{u})]$$
$$=\{\bigcap_{\epsilon'}(L-\epsilon'\bar{u})\}\cap[\bigcup_\epsilon(P_\epsilon-\epsilon\bar{u})]=\bigcap_{\epsilon'}(L-\epsilon'\bar{u})$$

(the last sign of equality owing to Lemma 2). But

$$\bigcap_{\epsilon'}(L-\epsilon'\bar{u})=\bigcap(L\mp\bar{u}_1\mp\cdots\mp\bar{u}_p),$$

where we shall consider all possible combinations of signs. And p successive applications of Lemma 1 show that this intersection is a non-empty convex body of minimal width not less than $l-h_1-\cdots-h_p$, which proves our theorem.

The theorem can be expressed briefly: The least 1-dimensional projection of a convex body is not greater than the sum of the 1-dimensional projections of its parts. It is worthwhile pointing out that the corresponding theorem for 2-dimensional projections is not valid. An example, maybe extremal, is the following: A regular tetrahedron is divided in two parts by a plane parallel to two opposite edges AB and CD and passing through the mid-points of the other four edges; the projection of each of the two parts in the directions AB and CD, respectively, is 1/4 of the minimal projection of the tetrahedron.

An interesting unsolved problem is the following: Is the sum of the relative widths always greater than or equal to 1, when a convex body is covered by strips (relative width of a strip = width of the strip divided by the width *in the same direction* of the convex body)?

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Jack Levine, *Collineations in generalized spaces.*
p. 455, reference 6. For "1928" read "1929."

J. L. Walsh, *On Rouché's theorem and the integral-square measure of approximation.*
p. 673, line 11. For "$\sum|az_n|^2$" read "$\sum|a_n|^2$."