LINEAR TRANSFORMATIONS ON OR ONTO
A BANACH SPACE

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We investigate here a simple property of linear transformations which are not necessarily bounded or closed or one-to-one, but whose domain or range is all of a Banach space.

**Theorem 1.** Let $T$ be a linear transformation from (all of) a Banach space $\mathcal{H}$ onto a normed vector space $Y$. Then there is a number $m > 0$ such that for any $x \in \mathcal{H}$ there exists a sequence $x_n \to x$ such that $\|Tx_n\| \leq m\|x\|$ and $\{Tx_n\}$ converges in the sense of Cauchy.

**Proof.** Let $C_n$ be the set of all $x \in \mathcal{H}$ such that $\|Tx\| \leq n$ ($n = 1, 2, 3, \ldots$). Then $\mathcal{H} = \bigcup_{n=1}^{\infty} C_n$. In virtue of the Baire category principle there is an integer $k$ such that $C_k$ contains a closed sphere, $S$, whose center and radius we denote by $x_0$ and $r$, respectively. Let $\|Tx_0\| = b$. Thus for each $z$ such that $\|z - x_0\| \leq r$ there exists a sequence $z_n \to z$ with $\|Tz_n\| \leq k$. Take $m = 2(k + b)/r$.

Now let $x \in \mathcal{H}$ be given. It suffices to consider $x \neq 0$, for if $x = 0$, the theorem is obvious if we use the sequence $x_n = 0$. Let $z = x_0 + rz/\|x\|$. Then $z_n \to z$ with $\|Tz_n\| \leq k$. Let $x_n' = (\|x\|/r)(z_n - x_0)$. Then $x_n' \to x$ and $\|Tx_n'\| \leq ((k + b)/r)\|x\| = (m/2)\|x\|$. Now we shall construct a sequence $\{x_n\}$ such that $\{Tx_n\}$ is, in addition, Cauchy convergent. For this we use the following lemma.

**Lemma.** For a given $x \in \mathcal{H}$, $x' \in \mathcal{H}$, there exists a sequence $u_n \to x$ with $\|Tx' - Tu_n\| \leq (m/2)\|x - x'\|$.

**Proof.** Applying the result already proved to the element $x - x'$ we have $x_n'' \to x - x'$ with $\|Tx_n''\| \leq (m/2)\|x - x'\|$. Let $u_n = x_n + x_n''$. Then $u_n \to x$ and $\|Tu_n - Tx\| = \|Tx_n''\| \leq (m/2)\|x - x'\|$, as asserted.

To complete the proof of the theorem take $n_1$ large enough so that $\|x - x_n'\| \leq \|x\|/2$ and $\|Tx_n'\| \leq (m/2)\|x\|$. Let $x_1 = x_n'$. By the lemma, $u_{n_1} \to x$ with $\|Tx_1 - Tu_{n_1}\| \leq (m/4)\|x\|$. Let $n_2$ be large enough so that $\|u_{n_2} - x\| \leq \|x\|/2^2$ and take $x_2 = u_{n_2}$. Again by the lemma, there exists $u_{n_2} \to x$ with $\|Tx_2 - Tu_{n_2}\| \leq m\|x\|/2^4$. Take $n_3$ large enough so that $\|u_{n_3} - x\| \leq (m/2^4)\|x\|$ and let $x_3 = u_{n_3}$. Continuing in this manner we have

$$\|Tx_i\| \leq \frac{m}{2} \|x\|,$$

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\[ \|Tx_1 - Tx_2\| \leq \frac{m}{4} \|x\|, \]
\[ \|Tx_3 - Tx_2\| \leq \frac{m}{2^3} \|x\|, \]
\[ \ldots \]
\[ \|Tx_n - Tx_{n-1}\| \leq \frac{m}{2^n} \|x\|, \quad \text{and} \quad \|x - x_n\| \leq \frac{m}{2^n}. \]

Thus \( x_n \to x \) and for \( p > q \geq 1 \):
\[ \|Tx_p - Tx_q\| \leq \|Tx_p - Tx_{p-1} + Tx_{p-1} - \ldots - Tx_{q+1} - Tx_q\| \leq m \|x\| (1/2^{q+1} + \ldots + 1/2^p) \leq m \|x\|/2^p, \]
which proves that the \( \{Tx_n\} \) converges in the sense of Cauchy. Finally
\[ \|Tx_n\| = \|Tx_n - Tx_{n-1} + Tx_{n-1} - \ldots - Tx_1 + Tx_1\| \]
\[ \leq \|Tx_n - Tx_{n-1}\| + \ldots + \|Tx_2 - Tx_1\| + \|Tx_1\| \]
\[ \leq m \|x\| \left( \frac{1}{2^n} + \ldots + \frac{1}{2^2} \right) + \frac{m}{2} \|x\| \leq m \|x\|. \]

It might be remarked that the closed graph theorem is an immediate corollary of this theorem (that is, if \( T \) is everywhere defined on a Banach space, then it is closed if and only if it is bounded). A further corollary is the fact that if \( T \) is everywhere defined and not closed, then for each \( x \in \mathbb{F} \) there exist three sequences \( x_n^{(1)} \to x, x_n^{(2)} \to x, x_n^{(3)} \to x \) with \( Fx_n^{(1)} \to y, Fx_n^{(2)} \to Tx, Fx_n^{(3)} \to y \) with \( \|y\| \leq m \|x\| \), where \( m \) is independent of \( x \).

**Theorem 2.** Let \( T \) be a linear transformation from a normed vector space \( X \) onto (all of) a Banach space \( Y \). Then there exists a number \( m > 0 \) such that for any \( y \in Y \), there exists a sequence \( y_n \to y \) with \( y_n = Tx_n, \|x_n\| \leq m \|y\| \), and \( \{x_n\} \) convergent in the sense of Cauchy.

**Proof.** The method is entirely analogous to that of Theorem 1 but we give the details. Let \( C_n \) be the set of all \( y \in Y \) such that \( y = Tx \) with \( \|x\| \leq n \) \( (n = 1, 2, 3, \ldots) \). Then \( Y \subseteq \sum_{n=1}^{\infty} C_n \). Hence there exists an integer \( k \) such that \( \overline{C_k} \) contains a sphere whose center and radius we denote by \( y_0 \) and \( r \) respectively. Say \( y_0 = Tx_0 \) with \( \|x_0\| = b \). Let \( m = 2(b+k)/r \). For any \( z \in Y \) such that \( \|z - y_0\| \leq r \) there exists \( z_n \to z \) with \( z_n = T\xi_n \) and \( \|\xi_n\| \leq b \). Let \( y \in Y \) be given. Clearly it suffices to consider \( y \neq 0 \). Let \( z = y_0 + (r/\|y\|)y \). Then \( z_n \) described above exists. Let \( y'_n = (\|y\|/r)(z_n - y_0) \). Then \( y'_n \to y, y'_n = Tx'_n \) (where \( x'_n = (\|y\|/r)(\xi_n - x_0) \)), and \( \|x'_n\| \leq ((b+k)/r)\|y\| = (m/2)\|y\| \).

Now we shall construct a sequence \( \{y_n\} \) such that \( \{y_n\} \) is, in
addition, Cauchy convergent. Again we use a lemma.

**Lemma.** For a given \( y \in Y \), \( y' = Tx' \in Y \) there exists a sequence \( v_n \rightarrow y \) with \( v_n = Tu_n \) and \( \|u_n - x'\| \leq (m/2)\|y - y'\| \).

**Proof.** Applying the result already established to the element \( y - y' \), we have \( y'' = y - y', y'' = Tx'' \), \( \|x''\| \leq (m/2)\|y - y'\| \). Set \( v_n = y' + y'' \). Then \( v_n \rightarrow y \), \( v_n = Tu_n \) (with \( u_n = x' + x'' \)), and \( \|u_n - x'\| = \|x''\| \leq (m/2)\|y - y'\| \), as asserted. To complete the proof of the theorem select \( n_1 \) large enough so that \( \|y_{n_1} - y\| \leq \|y\|/2 \). Let \( y_{n_1}' = y_1, x_{n_1}' = x_1 \). Then \( y_1 = Tx_1, \|x_1\| \leq (m/2)\|y\| \). Take \( n_2 \) large enough (by the lemma) so that \( \|v_{n_2} - y\| \leq \|y\|/4 \). \( v_{n_2} = Tu_{n_2} \), and \( \|u_{n_2} - x_1\| \leq (m/2)\|y - y_1\| \leq (m/4)\|y\| \). Let \( v_{n_2} = y_2, u_{n_2} = x_2 \). Take \( n_3 \) large enough so that \( \|v_{n_3} - y\| \leq \|y\|/2^2 \). \( v_{n_3} = Tu_{n_3} \), \( \|u_{n_3} - x_2\| \leq (m/2)\|y - y_2\| \leq (m/2^2)\|y\| \). Let \( v_{n_3} = y_3, u_{n_3} = x_3 \). Continuing in this manner we find a sequence \( y_n = Tx_n, \|y_n - y\| \leq \|y\|/2^n \), \( \|x_n - x_{n-1}\| \leq (m/2^n)\|y\| \). Thus \( y_n \rightarrow y \). For \( p > q \geq 1 \),

\[
\|x_p - x_q\| = \|x_p - x_{p-1} + x_{p-1} - \cdots + x_{q+1} - x_q\| \\
\leq m\|y\| \left( \frac{1}{2^p} + \cdots + \frac{1}{2^{q+1}} \right) \leq \frac{m\|y\|}{2^q}
\]

so that \( \{x_n\} \) converges in the sense of Cauchy. Finally

\[
\|x_n\| = \|x_n - x_{n-1} + x_{n-1} - \cdots + x_2 - x_1 + x_1\| \\
\leq m\|y\| \left( \frac{1}{2^n} + \cdots + \frac{1}{2^2} + \frac{1}{2} \right) \leq m\|y\|.
\]