LINEAR TRANSFORMATIONS ON OR ONTO A BANACH SPACE

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We investigate here a simple property of linear transformations which are not necessarily bounded or closed or one-to-one, but whose domain or range is all of a Banach space.

THEOREM 1. *Let T be a linear transformation from (all of) a Banach space $\mathfrak{X}$ onto a normed vector space Y. Then there is a number $m>0$ such that for any $x\in\mathfrak{X}$ there exists a sequence $x_n\to x$ such that $\|Tx_n\| \leqq m\|x\|$ and $\{Tx_n\}$ converges in the sense of Cauchy.*

PROOF. Let C_n be the set of all $x\in\mathfrak{X}$ such that $\|Tx\|\leqq n$ $(n=1, 2, 3, \cdots)$. Then $\mathfrak{X}=\sum_{n=1}^{\infty} C_n$. In virtue of the Baire category principle there is an integer k such that $\overline{C}_k$ contains a closed sphere, S, whose center and radius we denote by x_0 and r, respectively. Let $\|Tx_0\|=b$. Thus for each z such that $\|z-x_0\|\leqq r$ there exists a sequence $z_n\to z$ with $\|Tz_n\|\leqq k$. Take $m=2(k+b)/r$.

Now let $x\in\mathfrak{X}$ be given. It suffices to consider $x\neq 0$, for if $x=0$, the theorem is obvious if we use the sequence $x_n=0$. Let $z=x_0+rx/\|x\|$. Then $z_n\to z$ with $\|Tz_n\|\leqq k$. Let $x_n'=(\|x\|/r)(z_n-x_0)$. Then $x_n'\to x$ and $\|Tx_n'\|\leqq((k+b)/r)\|x\|=(m/2)\|x\|$. Now we shall construct a sequence $\{x_n\}$ such that $\{Tx_n\}$ is, in addition, Cauchy convergent. For this we use the following lemma.

LEMMA. *For a given $x\in\mathfrak{X}$, $x'\in\mathfrak{X}$, there exists a sequence $u_n\to x$ with $\|Tx'-Tu_n\|\leqq(m/2)\|x-x'\|$.*

PROOF. Applying the result already proved to the element $x-x'$ we have $x_n''\to x-x'$ with $\|Tx_n''\|\leqq(m/2)\|x-x'\|$. Let $u_n=x'+x_n''$. Then $u_n\to x$ and $\|Tu_n-Tx'\|=\|Tx_n''\|\leqq(m/2)\|x-x'\|$, as asserted.

To complete the proof of the theorem take n_1 large enough so that $\|x-x_{n_1}'\|\leqq\|x\|/2$ and $\|Tx_{n_1}'\|\leqq(m/2)\|x\|$. Let $x_1=x_{n_1}'$. By the lemma, $u_n^{(1)}\to x$ with $\|Tx_1-Tu_n^{(1)}\|\leqq(m/4)\|x\|$. Let n_2 be large enough so that $\|u_{n_2}^{(1)}-x\|\leqq\|x\|/2^2$ and take $x_2=u_{n_2}^{(1)}$. Again by the lemma, there exists $u_n^{(2)}\to x$ with $\|Tx_2-Tu_n^{(2)}\|\leqq m\|x\|/2^3$. Take n_3 large enough so that $\|u_{n_3}^{(2)}-x\|\leqq(m/2^3)\|x\|$ and let $x_3=u_{n_3}^{(2)}$. Continuing in this manner we have

$$\|Tx_1\| \leqq \frac{m}{2}\|x\|,$$

Received by the editors May 10, 1951.

$$\|Tx_1 - Tx_2\| \leqq \frac{m}{4}\|x\|,$$

$$\|Tx_3 - Tx_2\| \leqq \frac{m}{2^3}\|x\|,$$

$$\cdots\cdots\cdots\cdots,$$

$$\|Tx_n - Tx_{n-1}\| \leqq \frac{m}{2^n}\|x\|, \text{ and } \|x - x_n\| \leqq \frac{m}{2^n}\cdot$$

Thus $x_n \to x$ and for $p > q \geqq 1$: $\|Tx_p - Tx_q\| = \|Tx_p - Tx_{p-1} + Tx_{p-1} - \cdots + Tx_{q+1} - Tx_q\| \leqq m\|x\|\,(1/2^{q+1} + \cdots + 1/2^p) \leqq m\|x\|/2^q$, which proves that the $\{Tx_n\}$ converges in the sense of Cauchy. Finally

$$\begin{aligned}\|Tx_n\| &= \|Tx_n - Tx_{n-1} + Tx_{n-1} - \cdots - Tx_1 + Tx_1\| \\ &\leqq \|Tx_n - Tx_{n-1}\| + \cdots + \|Tx_2 - Tx_1\| + \|Tx_1\| \\ &\leqq m\|x\|\left(\frac{1}{2^n} + \cdots + \frac{1}{2^2}\right) + \frac{m}{2}\|x\| \leqq m\|x\|.\end{aligned}$$

It might be remarked that the closed graph theorem is an immediate corollary of this theorem (that is, if T is everywhere defined on a Banach space, then it is closed if and only if it is bounded). A further corollary is the fact that if T is everywhere defined and not closed, then for each $x \in \mathfrak{X}$ there exist three sequences $x_n^{(1)} \to x$, $x_n^{(2)} \to x$, $x_n^{(3)} \to x$ with $Tx_n^{(1)} \to \infty$, $Tx_n^{(2)} \to Tx$, $Tx_n^{(3)} \to y$ with $\|y\| \leqq m\|x\|$, where m is independent of x.

THEOREM 2. *Let T be a linear transformation from a normed vector space X onto (all of) a Banach space Y. Then there exists a number $m > 0$ such that for any $y \in Y$, there exists a sequence $y_n \to y$ with $y_n = Tx_n$, $\|x_n\| \leqq m\|y\|$, and $\{x_n\}$ convergent in the sense of Cauchy.*

PROOF. The method is entirely analogous to that of Theorem 1 but we give the details. Let C_n be the set of all $y \in Y$ such that $y = Tx$ with $\|x\| \leqq n$ $(n = 1, 2, 3, \cdots)$. Then $Y = \sum_{n=1}^{\infty} C_n$. Hence there exists an integer k such that $\overline{C}_k$ contains a sphere whose center and radius we denote by y_0 and r respectively. Say $y_0 = Tx_0$, with $\|x_0\| = b$. Let $m = 2(b+k)/r$. For any $z \in Y$ such that $\|z - y_0\| \leqq r$ there exists $z_n \to z$ with $z_n = T\xi_n$ and $\|\xi_n\| \leqq k$. Let $y \in Y$ be given. Clearly it suffices to consider $y \neq 0$. Let $z = y_0 + (r/\|y\|)y$. Then the z_n described above exists. Let $y_n' = (\|y\|/r)(z_n - y_0)$. Then $y_n' \to y$, $y_n' = Tx_n'$ (where $x_n' = (\|y\|/r)(\xi_n - x_0)$), and $\|x_n'\| \leqq ((k+b)/r)\|y\| = (m/2)\|y\|$.

Now we shall construct a sequence $\{y_n\}$ such that $\{y_n\}$ is, in

addition, Cauchy convergent. Again we use a lemma.

LEMMA. *For a given* $y \in Y$, $y' = Tx' \in Y$ *there exists a sequence* $v_n \to y$ *with* $v_n = Tu_n$ *and* $\|u_n - x'\| \leqq (m/2)\|y - y'\|$.

PROOF. Applying the result already established to the element $y - y'$, we have $y_n'' \to y - y'$, $y_n'' = Tx_n''$, $\|x_n''\| \leqq (m/2)\|y - y'\|$. Set $v_n = y' + y_n''$. Then $v_n \to y$, $v_n = Tu_n$ (with $u_n = x' + x_n''$), and $\|u_n - x'\| = \|x_n''\| \leqq (m/2)\|y - y'\|$, as asserted. To complete the proof of the theorem select n_1 large enough so that $\|y_{n_1}' - y\| \leqq \|y\|/2$. Let $y_{n_1}' = y_1$, $x_{n_1}' = x_1$. Then $y_1 = Tx_1$, $\|x_1\| \leqq (m/2)\|y\|$. Take n_2 large enough (by the lemma) so that $\|v_{n_2} - y\| \leqq \|y\|/4$, $v_{n_2} = Tu_{n_2}$, and $\|u_{n_2} - x_1\| \leqq (m/2)\|y - y_1\| \leqq (m/4)\|y\|$. Let $v_{n_2} = y_2$, $u_{n_2} = x_2$. Take n_3 large enough so that $\|v_{n_3} - y\| \leqq \|y\|/2^3$, $v_{n_3} = Tu_{n_3}$, $\|u_{n_3} - x_2\| \leqq (m/2)\|y - y_2\| \leqq (m/2^3)\|y\|$. Let $v_{n_3} = y_3$, $u_{n_3} = x_3$. Continuing in this manner we find a sequence $y_n = Tx_n$, $\|y_n - y\| \leqq \|y\|/2^n$, $\|x_n - x_{n-1}\| \leqq (m/2^n)\|y\|$. Thus $y_n \to y$. For $p > q \geqq 1$,

$$\begin{aligned}\|x_p - x_q\| &= \|x_p - x_{p-1} + x_{p-1} - \cdots + x_{q+1} - x_q\| \\ &\leqq m\|y\|\left(\frac{1}{2^p} + \cdots + \frac{1}{2^{q+1}}\right) \leqq \frac{m\|y\|}{2^q}\end{aligned}$$

so that $\{x_n\}$ converges in the sense of Cauchy. Finally

$$\begin{aligned}\|x_n\| &= \|x_n - x_{n-1} + x_{n-1} - \cdots + x_2 - x_1 + x_1\| \\ &\leqq m\|y\|\left(\frac{1}{2^n} + \cdots + \frac{1}{2^2} + \frac{1}{2}\right) \leqq m\|y\|.\end{aligned}$$

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