CONTINUITY OF AREA FOR HARMONIC SURFACES WITH BOUNDARIES OF UNIFORMLY BOUNDED LENGTH

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The object of this note is to supplement a paper by Young [1],[1] *On the isoperimetric ratio for a harmonic surface*, by answering in the affirmative a question raised by Morse and Tompkins [2]. Continuity of area as the harmonic surface S^* shrinks to a point is dealt with by Young by means of his main inequality. Here the same machinery deals with the more general case in which S^* approximates another surface S.

In what follows $r=|u+iv|$, D is the disc $r\leqq 1$, R its rim, and, in correspondence with a real number t where $0<t<1$, D^- is the concentric disc $r\leqq 1-t$ and R^+ is the annulus $1-t\leqq r\leqq 1$. We shall divide R^+ into n parts σ by the n segments λ in which it is met by the lines amp $(u+iv)=2\pi k/n$, where $-n/2<k\leqq n/2$. The suffixes t, or k, n, t in D^- and R^+, or in σ and λ, are understood to have been dropped.

S and S^* denote continuous vector functions $x(u, v)$ and $x^*(u, v)$ on D and it is assumed that $x^*(u, v)$ is the harmonic extension to D of a continuous vector function on R. We shall assume further that $(1-r)|x_u+ix_v|$ is bounded in D, and we denote by b its supremum; b^* is similarly defined in terms of S^* [and is finite since $x^*(u, v)$ is harmonic].

We write $d(S, S^*)$ and $\rho(S, S^*)$ respectively for the suprema in D of the expressions $|x(u,v)-x^*(u,v)|$ and $(1-r)|(x_u-x_u^*)+i(x_v-x_v^*)|$. Moreover, given any subset E of D whose boundary is a simple closed curve C, we write $\delta(E)$ for the diameter of the set of values $x(E)$ taken by $x(u, v)$ in E, and $A(E)$, $L(E)$ respectively for the area and boundary-length defined by the expressions

$$\iint_E \{x_u^2x_v^2-(x_ux_v)^2\}^{1/2}dudv, \qquad \int_C |dx|;$$

$\delta^*(E)$, $A^*(E)$, $L^*(E)$ denote the similar quantities for S^* and the argument E is omitted when $E=D$.

LEMMA 1. *For the sum of the angles α subtended at* $\cos\gamma+i\sin\gamma$ *by the n segments λ, we have the estimate $\sum\alpha<\pi$ provided that $t\leqq\eta(n)$ where $\eta(n)=[n(1+\log n)]^{-1}$.*

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[1] Numbers in brackets refer to the references at the end of the paper.

PROOF. We first estimate the angle α, subtended by the single segment $\lambda = AB$ on which amp $(u+iv)=\theta$, in terms of $\beta = |\theta-\gamma|/2$ where $\theta-\gamma$ is reduced mod 2π so that $2\beta \leqq \pi$. The diameter AC containing AB subtends a right angle at $M = \cos\gamma + i\sin\gamma$ and we have two expressions for the sine of the angle ϕ subtended by MC at B:

$$\begin{aligned} 2\sin\beta\sin\alpha/t &\equiv \overline{AM}\sin\alpha/\overline{AB} = \sin\phi \\ &= \overline{MC}\cos\alpha/\overline{BC} \equiv 2\cos\beta\cos\alpha/(2-t). \end{aligned}$$

Hence if $\beta \neq 0$ we find that $\alpha < \tan\alpha = t\cot\beta/(2-t) < t\cot\beta < t/\beta$; moreover, in any case, $\alpha \leqq \pi/2$.

To establish our lemma, in which we may suppose by circular symmetry that $0 \leqq \gamma \leqq \pi/n$, we have, on denoting by $\sum'$ and $\sum''$ sums for $-n/2 < k < 0$ and $0 < k \leqq n/2$,

$$\begin{aligned} \sum \alpha &< \frac{\pi}{2} + (\sum' + \sum'')\frac{t}{\beta} = \frac{\pi}{2} + t\sum'\left(\frac{\gamma}{2} - \frac{k\pi}{n}\right)^{-1} \\ &\quad + t\sum''\left(\frac{k\pi}{n} - \frac{\gamma}{2}\right)^{-1} \\ &\leqq \frac{\pi}{2} + t\sum'\left(-\frac{k\pi}{n}\right)^{-1} + t\sum''\left[\left(k - \frac{1}{2}\right)\frac{\pi}{n}\right]^{-1} \\ &\leqq \frac{\pi}{2} + \frac{2nt}{\pi}\left\{1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right\} \\ &< \frac{\pi}{2} + \frac{2t}{\pi}n(1+\log n) < \frac{\pi}{2}\left[1 + tn(1+\log n)\right] \leqq \pi. \end{aligned}$$

LEMMA 2. *For every S^* the inequality $t \leqq \eta(n)$ implies $\sum L^*(\sigma) \leqq 5L^*$.*

PROOF. Let $\Lambda = \sum \int_\lambda |dx^*|$. By Young's variant [**1**, p. 400, (3.2)] of F. Carlson's inequality [**3**], we have, since x^* is harmonic,

$$\Lambda \leqq \pi^{-1}\int_R (\sum \alpha)\,|dx^*| < \int_R |dx^*| = L^*,$$

provided that $t \leqq \eta(n)$. Our assertion now follows from the relation $\sum L^*(\sigma) = 2\Lambda + L^* + L^*(D^-)$ where the last term on the right cannot exceed $2L^*$ (by [**1**, p. 405 middle of page]; actually $L^*(D^-) \leqq L^*$).

LEMMA 3. *There is an absolute constant K and, given S and $\epsilon > 0$, a number $t = t_0(\epsilon, S) < \epsilon$, such that the inequality $d(S, S^*) < \epsilon$ implies $A^*(R^+) \leqq \epsilon K L^*$.*

PROOF. By uniform continuity of $x(u, v)$, there exist $n = n(\epsilon, S)$ and $t = t_0(\epsilon, S) \leqq \eta(n)$, so that $\delta(\sigma) < \epsilon$. This evidently insures $\delta^*(\sigma)$

$<3\epsilon$, while by Lemma 2, $\sum L^*(\sigma) \leqq 5L^*$. Consequently (applying Young's isoperimetric inequality $A \leqq KLd$ of [1] to the harmonic surface $x^*(u, v)$ on σ and summing),

$$A^*(R^+) = \sum A^*(\sigma) \leqq K \sum L^*(\sigma)\delta^*(\sigma) \leqq 15\epsilon KL^*,$$

which is of the asserted form, with $15K$ for K.

LEMMA 4. *There is an absolute constant K so that the inequality $\rho(S, S^*) < t^4$ implies $|A(D^-) - A^*(D^-)| \leqq Kt(b+b^*)^{3/2}$.*

PROOF. Since the quantities $|x_u + ix_v|/b$, $|x_u^* + ix_v^*|/b^*$, and $|(x_u - x_u^*) + i(x_v - x_v^*)|/\rho(S, S^*)$ do not exceed $(1-r)^{-1}$, the expression $U = x_u^2 x_v^2 - (x_u x_v)^2$ differs[2] from the corresponding expression U^* by at most $K^2(1-r)^{-4}\rho(S, S^*)(b+b^*)^3$ where K is an absolute constant. Since

$$\begin{aligned} |U^{1/2} - U^{*1/2}| &\leqq \{ |U^{1/2} - U^{*1/2}| (U^{1/2} + U^{*1/2}) \}^{1/2} \\ &= |U - U^*|^{1/2}, \end{aligned}$$

it follows that

$$\begin{aligned} |A(D^-) - A^*(D^-)| &\leqq \iint_{D^-} |U - U^*|^{1/2} du dv \\ &= 2\pi K t^2 (b + b^*)^{3/2} \int_0^{1-t} (1 - r)^{-2} r dr \end{aligned}$$

which implies our assertion, because

$$\int (1 - r)^{-2} r dr \leqq \int (1 - r)^{-2} dr = t^{-1} - 1 < t^{-1}.$$

LEMMA 5. *Given S and the constant N and given $\epsilon' > 0$, there exists $\epsilon'' = \epsilon''(S, N, \epsilon') > 0$ such that the relations $\rho(S, S^*) + d(S, S^*) < \epsilon''$, $L^* \leqq N$, $A < \infty$ together imply $|A - A^*| < \epsilon'$ for every harmonic S^*.*

PROOF. Clearly $\delta^* \leqq L^*$ and by an inequality of Schwarz [1, p. 398, (2.3)] $b^* \leqq 2\delta^*$. Moreover $b \leqq b^* + \rho(S, S^*)$, so that $b < \infty$. We may therefore suppose $N \geqq b$ without loss of generality (since b depends only on S) and we then have $b + b^* \leqq 3N$.

Given $\epsilon > 0$, we now determine $t = t(\epsilon, S) < t_0(\epsilon, S)$, where $t_0(\epsilon, S)$ is defined as in Lemma 3, so that

$$A(R^+) < \epsilon.$$

[2] $|x_u^2 x_v^2 - x_u^{*2} x_v^{*2}| = |x_v^2(x_u + x_u^*)(x_u - x_u^*) + x_u^{*2}(x_v + x_v^*)(x_v - x_v^*)| \leqq 2[(b + b^*)/(1-r)]^3 \cdot \rho(S, S^*)/(1-r)$. Similarly this expression majorizes $|(x_u^* x_v^*)^2 - (x_u x_v)^2|$.

This is possible since A is finite. Then the inequality $\rho(S, S^*) + d(S, S^*) < t^4$ implies on one hand by Lemma 3 (since $t^4 < t < \epsilon$)

$$A^*(R^+) \leqq KN\epsilon,$$

and on the other hand by Lemma 4

$$| A(D^-) - A^*(D^-) | < K\epsilon(3N)^{3/2}.$$

Hence by addition the quantity

$$| A - A^* | = | A(R^+) - A^*(R^+) + \{A(D^-) - A^*(D^-)\} |$$

is less than $\epsilon + KN\epsilon + K\epsilon(3N)^{3/2} = \epsilon'$ provided our hypotheses are satisfied and $\epsilon'' = t^4$ where $t = t(\epsilon, S)$. This completes the proof.

CONTINUITY THEOREM. *If S is harmonic and N fixed, then for harmonic S^* the relations $d(S, S^*) \to 0$ and $L^* \leqq N$ imply $A^* \to A$.*

PROOF. By semi-continuity of area we may suppose $A < \infty$. It is now sufficient to apply Lemma 5 with $3\epsilon''$ in place of ϵ'' since $\rho(S, S^*) \leqq 2d(S, S^*)$ by the inequality of Schwarz already referred to.

The theorem just proved answers the question raised by Morse and Tompkins [2, p. 828]. We do not know, however, whether the result remains true if $d(S, S^*)$ is replaced by the Fréchet distance of the harmonic surfaces S and S^*.

REFERENCES

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