A NOTE ON THE LOCATION OF THE ZEROS OF POLYNOMIALS

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Let \( p(z) = \sum_{j=0}^{n} a_j z^j \), and for all pairs \( a_j, a_{j+1} \) of successive nonzero coefficients, form the ratios

\[
(1) \quad r_j = (-1)^{j+1} \frac{a_j}{a_{j+1}}.
\]

Certain relationships exist between the location in the complex plane of the ratios (1) and the zeros of \( p(z) \).

Takahashi has proved\(^1\) that every sector of the plane, with vertex at the origin, which contains all the zeros of \( p(z) \) must also contain all the ratios (1).

The present note gives an answer to the converse question, that of finding restrictions on the location of the zeros of \( p(z) \) when all the ratios (1) are known to lie in a given sector of the plane.

**Theorem.** Let the ratios (1) all lie in the sector

\[
S: \quad \theta - \frac{\pi}{h} \leq \arg z \leq \theta + \frac{\pi}{h} \quad (h \geq n).
\]

Then all the zeros of \( p(z) \) lie in the sector

\[
S_0: \quad \theta - \frac{\pi}{n} - \frac{\pi}{h} \leq \arg z \leq \theta + \frac{\pi}{n} + \frac{\pi}{h},
\]

and this is the best bound possible.

Since the relative positions of the zeros of \( p(z) \) and the ratios (1) are not altered by rotations, we may, by a suitable preliminary rotation, make \( \theta = 0 \). We may also assume that the leading coefficient of \( p(z) \) is unity.

The theorem is proved first under the assumption that all the coefficients of \( p(z) \) are nonzero. We may then write

\[
p(z) = z^n - r_{n-1} z^{n-1} + r_{n-2} z^{n-2} - \cdots + (-1)^n (r_{n-1} \cdot \cdots \cdot r_0).
\]

If \( z_0 | e^{i(\theta + \alpha)} = - | z_0 | e^{i\alpha}, \text{ if } r_f \geq | r_f | e^{i\alpha} (\text{all } f), \text{ and if we let } \theta_n = 0 \text{ for }\)

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convenience, we have
\[ (-1)^n p(z_0) = \sum_{j=0}^{n} A_j e^{i(\alpha + \theta_n + \theta_{n-1} + \cdots + \theta_j)}, \]
where the \( A_j \) are real positive numbers. If all the ratios lie in \( S \), while \( z_0 \) lies outside \( S_0 \), we have
\[ |\theta_j| \leq \frac{\pi}{h} \quad (j = 0, \ldots, n - 1), \quad |\alpha| < \left( \frac{1}{n} - \frac{1}{h} \right) \pi. \]

We show that under these circumstances \( p(z_0) \neq 0 \).

We make the proof in two parts. First, if \( h \geq 2n \), we consider
\[ (-1)^n \{ \Re [p(z_0)] \cos k\alpha + \Im [p(z_0)] \sin k\alpha \} \]
\[ = (-1)^n \sum_{j=0}^{n} A_j \cos [(j - k)\alpha + \theta_n + \cdots + \theta_j], \]
where
\[ k = n - \frac{hn}{2(h - n)}. \]

If \( z_0 \) is a zero of \( p(z) \), the left side of this equality must be zero. But under our assumption that \( z_0 \) lies outside \( S_0 \) we can show that all the angles involved on the right side lie in \( |\arg z| \leq \pi/2 \), with the strict inequality holding at least once, whence the right side is positive. For \( k \leq j \leq n \), we have
\[ |(j - k)\alpha + \theta_n + \cdots + \theta_j| \leq \left[ \frac{(j - k)}{n} \left( \frac{1}{n} - \frac{1}{h} \right) + \frac{n - j}{h} \right] \pi \]
\[ = \left[ \frac{j}{n} + \frac{2(n - j)}{h} - \frac{1}{2} \right] \pi \]
\[ \leq \left[ \frac{j}{n} + \frac{2(n - j)}{2n} - \frac{1}{2} \right] \pi = \frac{\pi}{2}. \]

For \( 0 \leq j < k \), we have
\[ |(j - k)\alpha + \theta_n + \cdots + \theta_j| < \left[ (k - j) \left( \frac{1}{n} - \frac{1}{h} \right) + \frac{n}{h} \right] \pi \]
\[ \leq \left[ k \left( \frac{1}{n} - \frac{1}{h} \right) + \frac{n}{h} \right] \pi = \frac{\pi}{2}. \]

When \( n \leq h < 2n \), we proceed similarly to consider the expression.
\((-1)^n \{ \Re [\varphi(z_0)] \cos (\theta_n + \cdots + \theta_{k+1} + \mu \theta_k) \\
+ \Im [\varphi(z_0)] \sin (\theta_n + \cdots + \theta_{k+1} + \mu \theta_k) \}\)
\[= (-1)^n \left\{ \sum_{j=k+2}^{n} A_j \cos \left( j \alpha - \theta_{j-1} - \cdots - \theta_{k+1} - \mu \theta_k \right) \right. \\
+ A_{k+1} \cos \left( (k + 1) \alpha - \mu \theta_k \right) \\
\left.+ \sum_{j=0}^{k} A_j \cos \left( j \alpha + \theta_k + \cdots + \theta_j - \mu \theta_k \right) \right\}, \]
where $$k = \left\lfloor \frac{h}{2} \right\rfloor, \quad \mu = \left\lfloor \frac{h}{2} \right\rfloor - \frac{h}{2} + 1.$$ For $$k+2 \leq j \leq n$$, we have
\[
\left| j \alpha - \theta_{j-1} - \cdots - \theta_{k+1} - \mu \theta_k \right| < \left[ j \left( \frac{1}{n} - \frac{1}{h} \right) + \frac{j - k - 1 + \mu}{h} \right] \pi \\
= \left[ \frac{j}{n} - \frac{1}{2} \right] \pi \leq \left[ \frac{n}{n} - \frac{1}{2} \right] \pi = \frac{\pi}{2}. \]
For the "middle term," we have
\[
\left| (k + 1) \alpha - \mu \theta_k \right| < \left[ (k + 1) \left( \frac{1}{n} - \frac{1}{h} \right) + \frac{\mu}{h} \right] \pi \\
= \left[ \frac{k + 1}{n} - \frac{1}{2} \right] \pi \leq \left[ \frac{n}{n} - \frac{1}{2} \right] \pi = \frac{\pi}{2}. \]
For $$0 \leq j \leq k$$, we have
\[
\left| j \alpha + \theta_k + \cdots + \theta_j - \mu \theta_k \right| \leq \left[ j \left( \frac{1}{n} - \frac{1}{h} \right) + \frac{k - j + 1 - \mu}{h} \right] \pi \\
= \left[ \frac{j}{n} - \frac{2j + 1}{h} \right] \pi \\
\leq \left[ \frac{j}{h/2} - \frac{2j + 1}{h} \right] \pi = \frac{\pi}{2}. \]
This finishes the proof for polynomials with nonvanishing coefficients.
To show that our theorem holds for a polynomial \( \varphi(z) \) with one or more zero coefficients, we consider the polynomial \( q(z) \) obtained by
inserting terms into \( p(z) \) as follows: Into each gap, where \( a_j, a_{j+s} \) \((s>1)\) are two consecutive nonvanishing coefficients, we insert the terms

\[
\epsilon \left[ -z^{j+s-1} + z^{j+s-2} - \cdots + (-1)^{r-1}z^{j+1} \right] \quad (\epsilon \neq 0).
\]

Now we may choose \( \arg \epsilon \) in such a way that the ratios for \( q(z) \) are contained in the sector \( S \) which contains those of \( p(z) \). For, corresponding to each gap where we have inserted terms, \( p(z) \) has the one ratio \( r = (-1)^s a_j/a_{j+s} \), while \( q(z) \) has \( s-2 \) ratios equal to 1, which automatically lie in \( S \), and the two ratios \( r' = \epsilon/a_{j+s} \) and \( r'' = (-1)^s a_j/\epsilon \).

But \( r = r' r'' \), so that by choosing 2 \( \arg \epsilon = \arg (-1)^s + \arg a_{j+s} + \arg a_j \), we have \( \arg r' = \arg r'' = (\arg r)/2 \). The ratios for \( q(z) \) do now lie in \( S \). Furthermore, since the zeros of a polynomial are continuous functions of its coefficients, we may, by taking \( |\epsilon| \) sufficiently small, make the zeros of \( q(z) \) arbitrarily near to those of \( p(z) \). But \( q(z) \) is a polynomial with nonvanishing coefficients, whence its zeros are known to lie in the closed sector \( S_0 \). Thus it is clear that those of \( p(z) \) must also lie in \( S_0 \).

The sector \( S_0 \) cannot be improved. For \( p(z) = z^n + (-1)^n \) has a zero on \( \arg z = \pi - \pi/n \); and the zeros of \( P(z) = p(z) + \epsilon \sum_{k=1}^{n-1} (-1)^k z^{n-k} \) \((\epsilon>0)\), with ratios \( \epsilon, 1, 1/\epsilon \) on the positive real axis, are arbitrarily near to those of \( p(z) \) for sufficiently small \( \epsilon \). The rotation \( z = e^{-i\pi/h} w \) furnishes the polynomial \( Q(w) = P(e^{-i\pi/h} w) \). Both the ratios and the zeros of \( Q(w) \) are obtained from those of \( P(z) \) by a rotation through \( \pi/h \). Hence there is a ratio of \( Q(w) \) on the upper boundary of \( S \), while there is a zero of \( Q(w) \) arbitrarily near to the upper boundary of \( S_0 \). Clearly there is a similar polynomial \( R(w) = P(e^{i\pi/h} w) \), whose ratios lie in \( S \) and which has a zero arbitrarily close to the lower boundary of \( S_0 \).

**Corollary 1.** If all the ratios lie in \( S \), then all the zeros of the \( k \)th derivative of \( p(z) \) lie in

\[
\theta - \pi \left( \frac{n-k-1}{n-k} + \frac{1}{h} \right) \leq \arg z \leq \theta + \pi \left( \frac{n-k-1}{n-k} + \frac{1}{h} \right).
\]

**Corollary 2.** If all the ratios lie in \( S \), then all the \( k \)-fold zeros of \( p(z) \) lie in

\[
\theta - \pi \left( \frac{n-k}{n-k+1} + \frac{1}{h} \right) \leq \arg z \leq \theta + \pi \left( \frac{n-k}{n-k+1} + \frac{1}{h} \right).
\]