ON AN EQUIVALENT DEFINITION OF THE TRANSFINITE DIAMETER

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Let

$$f(\zeta) = a\zeta + a_0 + a_1/\zeta + \cdots, \qquad a > 0, \tag{1}$$

map the exterior of the unit circle C onto the exterior of a simply-connected domain B. It is part of a general theory (see e.g. [3]) that the quantity $1/a$ can be identified with the transfinite diameter of B. In particular, let $T_n(z)$ be the polynomial of degree n with leading coefficient 1 whose L^∞ norm over B, i.e. the maximum of whose absolute value over $\overline{B}$, is minimum. Then

$$\lim_{n\to\infty} (M_n)^{1/n} = a.$$

The object of this paper is to show that the last result holds true in the case of domains with a Jordan boundary if we replace the minimal polynomials in the L^∞ metric by those in any L^p metric, $p \geqq 2$. Let $Q_n^{(p)}(z)$ be the polynomial minimizing

$$\left(\iint_B |Q_n(z)|^p dxdy\right)^{1/p} \tag{2}$$

among all polynomials of degree n with leading coefficient 1, and let $\lambda_n^{(p)}$ be the L^p norm of $Q_n^{(p)}(z)$, i.e. the value of (2) for $Q_n = Q_n^{(p)}$. Then $\lim_{n\to\infty} (\lambda_n^{(p)})^{1/n} = a$. After a preliminary lemma substantially due to Carleman [2], we shall prove the result first for $p=2$, and then for $2 < p < \infty$.

LEMMA. *Let B have an analytic boundary. Then* $\lim_{n\to\infty} (\lambda_n^{(2)})^{1/n} = a$.

PROOF. Since $T_n(z)$ is a competing polynomial in the L^2 minimum problem,

$$(\lambda_n^{(2)})^2 \leqq \iint_B |T_n(z)|^2 dxdy \leqq M_n^2 \cdot A, \tag{3}$$

where A is the area of B, and

$$\limsup_{n\to\infty} (\lambda_n^{(2)})^{1/n} \leqq \lim (M_n)^{1/n} = a. \tag{4}$$

On the other hand, since B has an analytic boundary, the function

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$f(\zeta)$ can be continued to be analytic and schlicht up to a circle γ_{ρ_1}: $|\zeta| = \rho_1 < 1$. Let $\rho_1 < \rho < 1$ and let D be the annulus bounded by γ_ρ and γ_1; furthermore, let $f(D)$ be its image in B. Then

$$\iint_B |Q_n^{(2)}(z)|^2 dxdy \geqq \iint_{f(D)} |Q_n^{(2)}(z)|^2 dxdy$$

(5)

$$= \iint_D |Q_n^{(2)}[f(\zeta)]f'(\zeta)|^2 d\xi d\eta.$$

Now $Q_n^{(2)}(z) = z^n + c_1 z^{n-1} + \cdots$, $f(\zeta) = a\zeta + a_0 + a_1/\zeta + \cdots$; hence

$$Q_n^{(2)}[f(\zeta)]f'(\zeta) = a^{n+1}\zeta^n + \sum_{j=1}^{\infty} b_j \zeta^{n-j}$$

and

$$\begin{aligned}(\lambda_n^{(2)})^2 &\geqq \iint_D \left| a^{n+1}\zeta^n + \sum_{j=1}^{\infty} b_j \zeta^{n-j} \right|^2 d\xi d\eta \\ &= 2\pi a^{2n+2} \int_\rho^1 r^{2n+1} dr + 2\pi \sum_{j=1}^{\infty} |b_j|^2 \int_\rho^1 r^{2n-2j+1} dr \qquad (6) \\ &\geqq \frac{\pi a^{2n+2}}{n+1}(1 - \rho^{2n+2}).\end{aligned}$$

Since $\rho < 1$, it follows that $\rho^{2n+2} \to 0$, and $\liminf_{n\to\infty} (\lambda_n^{(2)})^{1/n} \geqq a$. By comparing (4) and (6) we see that $\lim_{n\to\infty} (\lambda_n^{(2)})^{1/n}$ exists and equals a.

THEOREM 1. *Let B be a simply-connected domain with a Jordan boundary. Then* $\lim_{n\to\infty} (\lambda_n^{(2)})^{1/n}$ *exists and equals a.*

PROOF. Let $B \supset B'$. Then if $Q_n^{(2)}(z)$ is the L^2 minimal polynomial over B,

$$(7)\quad [\lambda_n^{(2)}(B)]^2 = \iint_B |Q_n(z)|^2 dxdy \geqq \iint_{B'} |Q_n(z)|^2 dxdy \geqq [\lambda_n^{(2)}(B')]^2.$$

Hence

$$(8)\qquad \limsup_{n\to\infty} [\lambda_n^{(2)}(B)]^{1/n} \geqq \limsup_{n\to\infty} [\lambda_n^{(2)}(B')]^{1/n};$$

the same is true for lim inf, and lim if it exists. Thus, we see that $\limsup_{n\to\infty} [\lambda_n^{(2)}(B)]^{1/n}$ and $\liminf_{n\to\infty} [\lambda_n^{(2)}(B)]^{1/n}$ are increasing set functions. Now an arbitrary domain can be approximated from the exterior by domains with analytic boundaries in such a way that the respective $a(B)$'s are arbitrarily close to each other; one needs only to take level lines of the exterior mapping function. In the case of a domain with a Jordan boundary, it follows from the Carathéodory theory [1] that this is also true for interior approximation. Since the transfinite diameter is also an increasing set function, it follows by approximation that $\lim (\lambda_n^{(2)})^{1/n}$ exists and equals a in the case of an arbitrary Jordan domain.

THEOREM 2. *Let $Q_n^{(p)}(z)$ be the polynomial of degree n with leading coefficient* 1 *minimizing*

$$\left(\iint_B |Q_n(z)|^p dxdy\right)^{1/p} \tag{9}$$

and let $\lambda_n^{(p)}$ be the corresponding minimum value of (9). *Then for any $p>2$, $\lim_{n\to\infty} (\lambda_n^{(p)})^{1/n}$ exists and equals a.*

PROOF. For any $f(z)\in L^2$ and L^p, $p>2$, we have

$$\left(\iint_B |f(z)|^2 dxdy\right)^{1/2} \leqq \left(\iint_B |f(z)|^p dxdy\right)^{1/p}. \tag{10}$$

Therefore

$$\lambda_n^{(2)} \leqq \left(\iint_B |Q_n^{(p)}(z)|^2 dxdy\right)^{1/2} \leqq \left(\iint_B |Q_n^{(p)}(z)|^p dxdy\right)^{1/p} = \lambda_n^{(p)}$$

and hence

$$a = \lim_{n\to\infty} (\lambda_n^{(2)})^{1/n} \leqq \liminf_{n\to\infty} (\lambda_n^{(p)})^{1/n}. \tag{11}$$

On the other hand,

$$(\lambda_n^{(p)})^p \leqq \iint_B |T_n(z)|^p dxdy \leqq A[M_n]^p,$$

where A is the area of B. Hence

$$\limsup_{n\to\infty} (\lambda_n^{(p)})^{1/n} \leqq \lim (M_n)^{1/n} = a. \tag{12}$$

Combining (11) and (12), we obtain that $\lim_{n\to\infty} (\lambda_n^{(p)})^{1/n}$ exists and equals a.

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