AN OPERATORIAL CHARACTERIZATION OF CERTAIN PARTITION POLYNOMIALS

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Introduction. Let \( I_n \) (\( n = 1, 2, 3, \ldots \)) be the initial arithmetic interval consisting of the first \( n \) natural numbers \( 0, \ldots, n - 1 \). A sequence \( (N_1, \ldots, N_x) \) of \( x \) sets of natural numbers is called an \( x \)-partition of \( I_n \) if \( I_n \) is the direct sum of the \( N \)'s. I have shown elsewhere\(^2\) that the number \( f_n(x) \) of \( x \)-partitions of \( I_n \) satisfies the recursion relation

\[
xf_n(x) = 1 + (x - 1) \sum_{d|n} f_d(x),
\]

the summation extending over all divisors \( d \) of \( n \). In the above context \( x \) must, of course, be a positive natural number. It is clear, however, that (1) uniquely determines \( f_n(x) \) as a polynomial in \( x \) with integer coefficients.

In this note we evaluate the partition polynomial \( f_n(x) \) explicitly. This can easily be done by the usual combinatorial methods of multiplicative number theory, and we shall first record one expression for \( f_n(x) \) obtained in this fashion. However, we are here mainly interested in the following operatorial characterization of \( f_n(x) \). Let \( D \) be the operator of formal differentiation with respect to \( x \) acting on the ring \( R \) of polynomials in the indeterminate \( x \) over the field of rational numbers; and let \( F_k \) be the operator

\[
F_k = -1 \cdot x^k D^k (x - 1)^k
\]

\((k \geq 0)\) (\(= \) denotes operator identity). We shall show that the operators \( F_k \) commute among themselves and that

\[
f_n(x) = F_{k_1} \cdots F_{k_m}[1]
\]

(operands are enclosed in brackets) where

\[
n = p_1^{k_1} \cdots p_m^{k_m}
\]

is a prime factorization of \( n \).

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\(^2\) Partitioning an arithmetic interval, not yet published.
Combinatorial solution. Given \( f_n(x) \) satisfying (1), let
\[
\phi = \phi(x, s) = \sum_{n=1}^{\infty} f_n(x) n^{-s}
\]
be the generating function of the \( f^n \)'s. Then (1) may be written in the form
\[
x\phi = \xi + (x - 1)\xi\phi
\]
where \( \xi = \xi(s) \) is the zeta function, so
\[
\phi = \frac{1}{1 - (1 - \xi^{-1})x} = \sum_{k=0}^{\infty} (1 - \xi^{-1})^k x^k.
\]
Separating out the coefficient of \( n^{-s} \) in this formula, we obtain the following explicit evaluation of \( f_n(x) \):
\[
f_n(x) = \sum_{k=0}^{\infty} c_k(n) x^k,
\]
where \( c_k(n) = \sum_{h=0}^{k} (-1)^h C_{k,h} \mu_h(n) \)
\[
\mu_h(n) = (-1)^{k_1 + \cdots + k_m} C_{h,k_1} \cdots C_{h,k_m},
\]
where \( n \) has the prime factorization (4); the \( \mu_h(n) \)'s are the generalized Moebius functions generated by \( \xi^{-h} \). The partition polynomial \( f_n(x) \) has degree \( k_1 + \cdots + k_m \) and the highest power of \( x \) dividing it is \( \max(k_1, \ldots, k_m) \).

Operatorial solution. We define the operator \( L \) as follows:\(^3\) it is to be a linear operator which operates on the basic polynomial operands \( x^i \) \((i \geq 0)\) of \( R \) with the result:
\[
(5) \quad L[x^i] = \begin{cases} 0 & (i = 0), \\ x^{i-1} & (i \geq 1). \end{cases}
\]
Therefore
\[
(6) \quad Lx = 1,
\]
so \( L \) is a left (but not a right) inverse of the operator \( x \). However, since

\(^3\) Remark. Each of the operators \( D \) and \( L \) can be expressed in terms of the other and the operator \( x \) in an infinite operator series as follows \( D = (1 + xL + x^2L^2 + \cdots)L \), \( L = (1 - (1/2!)xD + (1/3!)x^2D^2 + \cdots)D \). No question of convergence is involved here, for all except finitely many terms produce 0 when operating on an element of \( R \).
(7) \[ x^i L^k[x^i] = \begin{cases} 0 & (i < k), \\ x^i & (i \geq k), \end{cases} \]

\( L \) can under suitable conditions act like a right inverse of \( x \). For example, it is clear from (7) that

(8) \[ D^k x x^L = D^k x^{b - h} \quad (h \leq k). \]

Now consider the operators

(9) \[ E_k = \frac{1}{k!} D^k x^k \quad (k \geq 0). \]

According to (8) we have

(10) \[ D^k = D^k x x^L = k! E_k x^k \quad (k \geq 0), \]

so

\[
\begin{align*}
F_k &= \frac{1}{k!} x^{k} D^k (x - 1)^k \\
&= x^{k} E_k x^k (x - 1)^k \\
&= x^{k} E_k (1 - L)^k 
\end{align*}
\]

by repeated application of (6).

We now derive a relation on the \( E \)'s which will induce a relation on the \( F \)'s. From Leibnitz's formula or by induction we get

\[ D^k x - xD^k \equiv kD^{k-1} \quad (k \geq 1), \]

whence by (8), (9), and (10)

\[
\begin{align*}
k!(E_k - xE_k x^L) &= D^k x^k - xD^k x^L \\
&= (D^k x - xD^k)x^{k-1} \\
&= kD^{k-1}x^{k-1} \\
&= k!E_{k-1} 
\end{align*}
\]

Therefore according to (11)

\[
\begin{align*}
F_k - xF_k x^L &= x^{k} E_k (1 - L)^k - x^{k} xE_k x^L (1 - L)^k \\
&= x^{k} (E_k - xE_k x^L) (1 - L)^k \\
&= x^{k} E_{k-1} (1 - L)^k \\
&= xF_{k-1} (1 - L) \quad (k \geq 1),
\end{align*}
\]

which can be written in the form
(12) \((x - 1)F_k \equiv x(F_k - F_{k-1})(1 - L)\) \((k \geq 1)\).

Let us associate operators \(F^*\) to the operators \(F\) as follows:

\[ F_0^* \equiv F_0(=1), \]
\[ F_k^* \equiv F_k - F_{k-1} \quad (k \geq 1), \]

whereupon

\[ F_k \equiv \sum_{k=0}^{k} F_k^* \quad (k \geq 0). \]

By a complex we shall mean an operator \(C\) formed by composition of \(F\)'s:

\[ C \equiv F_{k_1} \cdots F_{k_m} \]

and we put

\[ C^* \equiv F_{k_1}^* \cdots F_{k_m}^*, \]

so that for any two complexes \(A\) and \(B\) we have

\[ (AB)^* \equiv A^*B^*. \]

The number of positive \(k\)'s occurring in the expression (15) for the complex \(C\) will be called the length of (that expression of) \(C\). Clearly, the identity operator \(1\) is the only complex of length \(0\).

We now show that

\[ (x - 1)C \equiv xC^*(1 - L) \quad \text{(length } C \geq 1) \]

for any complex \(C\) of positive length by induction on the length \(l\) of \(C\). Formula (12) together with definition (13) assures us that (18) is valid for any complex \(C\) of length \(l=1\). Assume (18) true for any complex of positive length less than \(l\) with \(l>1\): we are to prove (18) for \(C\) of length \(l\). Since \(l>1\), \(C\) can be factored into two complexes \(A\) and \(B\) each of positive length less than \(l\) so that \(C=AB\). Therefore using (6) and (17) and the fact that both \(A\) and \(B\) satisfy (18) we have

\[ (x - 1)AB \equiv xA^*(1 - L)B \]
\[ \equiv xA^*B - xA^*L(xB - xB^*(1 - L)) \]
\[ \equiv xA^*B - xA^*B + xA^*B^*(1 - L) \]
\[ \equiv x(AB)^*(1 - L). \]
This shows that $C$ satisfies \( (18) \), so by induction \( (18) \) is valid for all complexes $C$ of positive length.

We are now in a position to verify that the operators $F$ commute with one another over $R$. It is clear from the rearranged form

\[
C = x(C - C^*) + xC^*L
\]

of formula \( (18) \), applied to the complex $C = F_h F_k$, that the commutativity of the $F$'s follows by induction both on $h + k$ and on the degree of the polynomial operand in virtue of the degree reducing property of the operator $L$.

Let us finally consider the operator

\[
S = x \sum_{h_1=0}^{k_1} \cdots \sum_{h_m=0}^{k_m} (F_{h_1} \cdots F_{h_m})^*(1 - L).
\]

We see from \( (16) \) and \( (14) \) that

\[
(19) \quad S = x F_{k_1} \cdots F_{k_m} (1 - L).
\]

On the other hand, formula \( (18) \) applies to every term of the sum $S$ except the single term of length 0, wherein all $h$'s are 0, so

\[
(20) \quad S - x(1 - L) = (x - 1) \sum_{h_1=0}^{k_1} \cdots \sum_{h_m=0}^{k_m} F_{h_1} \cdots F_{h_m} - (x - 1).
\]

By eliminating $S$ between \( (19) \) and \( (20) \) we obtain the following relation for complexes:

\[
x F_{k_1} \cdots F_{k_m} = 1 + (x - 1) \sum_{h_1=0}^{k_1} \cdots \sum_{h_m=0}^{k_m} F_{h_1} \cdots F_{h_m}
\]

\[+ x (F_{k_1} \cdots F_{k_m} - 1)L.
\]

Since $L[1] = 0$, we conclude from this relation that the functions $f_n(x)$ defined by \( (3) \) and \( (4) \) satisfy \( (1) \)—as was to be proved.

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\footnote{It can be shown more generally and independently of the particular nature of the ring $R$ of operands that the operators $F_h = a^{h_1} b^{h_2} D^{h_3} h^{h_4} k^{h_5}$ ($h, k \geq 0$) commute among themselves, where $a$ and $b$ are polynomials in $x$ of at most first degree.}