NOTE ON BOUNDS FOR DETERMINANTS WITH DOMINANT PRINCIPAL DIAGONAL

A. M. OSTROWSKI

1. Let $A = |a_{\mu\nu}|$ $(\mu, \nu = 1, \cdots, n)$ be a determinant and put

$$\alpha_{\mu\nu} = |a_{\mu\nu}| \ (\mu, \nu = 1, \cdots, n), \ \alpha_{\mu\mu} \neq 0 \ (\mu = 1, \cdots, n),$$

$$\sum_{\nu=1}^{n} \alpha_{\mu\nu} - \alpha_{\mu\mu} = s_\mu = \sigma_\mu \alpha_{\mu\mu}. \tag{1}$$

Under the hypothesis that all σ_μ are less than 1, I proved (1937)[1] that

$$|A| \geqq \prod_{\mu=1}^{n} (\alpha_{\mu\mu} - s_\mu). \tag{2}$$

This inequality provides a simple lower bound for $|A|$ which is however of the wrong order if all s_μ tend to 0; for in this case the difference $A - a_{11}a_{22} \cdots a_{nn}$ is at least of the *second* order in the $\alpha_{\mu\nu}$ $(\mu \neq \nu)$.

2. Some months later[2] I gave another estimate for $|A|$. Suppose that the σ_μ are arranged decreasingly:

$$\sigma_{m_1} \geqq \sigma_{m_2} \geqq \cdots \geqq \sigma_{m_n}, \tag{3}$$

then we have:

$$\begin{aligned} (\alpha_{m_1m_1}\alpha_{m_2m_2} + s_{m_1}s_{m_2})(\alpha_{m_3m_3}\alpha_{m_4m_4} + s_{m_3}s_{m_4}) \cdots \geqq |A| \\ \geqq (\alpha_{m_1m_1}\alpha_{m_2m_2} - s_{m_1}s_{m_2})(\alpha_{m_3m_3}\alpha_{m_4m_4} - s_{m_3}s_{m_4}) \cdots , \end{aligned} \tag{4}$$

where if n is odd, the last factors to the left and to the right in (4) are resp. $\alpha_{m_{n-2}m_{n-2}}\alpha_{m_{n-1}m_{n-1}} \pm s_{m_{n-2}}s_{m_{n-1}}$. All other couples $\alpha_{m_\nu m_\nu}$, s_{m_ν} are paired together as shown in (4).

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[1] A. Ostrowski, *Sur la détermination des bornes inférieures pour une classe des déterminants*, Bull. Sci. Math. (2) vol. 61 (1937) pp. 19–32. The fact that, if all σ_μ are less than 1, A does not vanish is due to Hadamard and others.

[2] A. Ostrowski, *Ueber die Determinanten mit überwiegender Hauptdiagonale*, Comment. Math. Helv. vol. 10 (1937) pp. 69–96. I use this opportunity to mention the following misprints in this paper: p. 70, l. 9, f.a., read $|h_{\mu\mu}|$ instead of $h_{\mu\mu}$; p. 73, formula (13), read $|h_{\mu\mu}|$ instead of $h_{\mu\mu}$ and $\sum_{\nu=1, \nu\neq\mu}^{n}$ instead of $\sum_{\mu=1, \mu\neq\nu}^{n}$; p. 73, l. 4, f.b., read 1881 instead of 1899; p. 76, the right-side product-sign in the formula (18) is to be dropped; p. 86, in the formula (11, 1) read on the left side $m_{\mu\mu}y_\mu$ instead of y_μ, and on the right side M instead of 1; p. 96, l. 7, f.b., read s_1/s_2 instead of s_2/s_1.

This result gives not only the best order of $A - a_{11} \cdots a_{nn}$, but is also "the best" in this sense, that both bounds *can be attained* for any given sets of positive $\alpha_{11}, \cdots, \alpha_{nn}$ and of non-negative $s_1, \cdots, s_n$. Besides, for the validity of (4) it is already sufficient that $\sigma_{m_1}\sigma_{m_2} < 1$, while σ_{m_1} can be greater than 1.

3. In two recent publications G. B. Price[3] and R. Oeder[4] suggested a new approach to the problem in using the expressions

$$r_\mu = \sum_{\nu=\mu+1}^{n} \alpha_{\mu\nu}, \qquad l_\mu = \sum_{\nu=1}^{\mu-1} \alpha_{\mu\nu}. \tag{5}$$

G. B. Price showed that (2) can be improved to the inequality

$$\prod_{\mu=1}^{n} (\alpha_{\mu\mu} + r_\mu) \geqq |A| \geqq \prod_{\mu=1}^{n} (\alpha_{\mu\mu} - r_\mu) \tag{6}$$

under the assumption that all σ_μ are less than 1, while R. Oeder gives, under the assumption that all σ_μ are less than or equal to 1, the inequality

$$|A| \geqq \alpha_{11} \prod_{\mu=2}^{n} (\alpha_{\mu\mu} - l_\mu). \tag{6'}$$

However, the very remarkable inequalities (6) and (6′) can still be improved in such a way that $A - a_{11}a_{22} \cdots a_{nn}$ becomes of the *second* order and the l_μ are used together with the r_μ. Put

$$\sigma = \max_\mu \sigma_\mu. \tag{7}$$

Then we have for an arbitrary index k from $1, \cdots, n$:

$$\begin{aligned} &\alpha_{kk} \prod_{\mu=1}^{k-1} (\alpha_{\mu\mu} + \sigma r_\mu) \prod_{\mu=k+1}^{n} (\alpha_{\mu\mu} + \sigma l_\mu) \geqq |A| \\ &\qquad \geqq \alpha_{kk} \prod_{\mu=1}^{k-1} (\alpha_{\mu\mu} - \sigma r_\mu) \prod_{\mu=k+1}^{n} (\alpha_{\mu\mu} - \sigma l_\mu) \end{aligned} \tag{8}$$

under the assumption that all σ_μ are less than or equal to 1. In (8), σ can be replaced by

$$t_\mu = \max_{\kappa \neq \mu} \sigma_\kappa. \tag{9}$$

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[3] G. B. Price, *Bounds for determinants with dominant principal diagonal*, Proceedings of the American Mathematical Society vol. 2 (1951) pp. 497–502.

[4] R. Oeder, Amer. Math. Monthly vol. 58 (1951) p. 37, problem E. 949.

4. Still, if we wish to obtain the best possible estimates of $|A|$, it is better not to use any of the formulae (8), but to avail oneself repeatedly of the following inequality in which A_m is the principal minor of A corresponding to a_{mm}:

$$(10) \qquad (\alpha_{mm} + t_m s_m)\,|A_m| \geqq |A| \geqq (\alpha_{mm} - t_m s_m)\,|A_m|.$$

This inequality is valid if we have

$$(11) \qquad t_m \leqq 1, \qquad t_m \sigma_m < 1.$$

We have even, under the hypothesis (11), the following a little more general relation

$$(10') \qquad A = A_m(a_{mm} + \theta_m t_m s_m), \qquad |\theta_m| \leqq 1.$$

To obtain (8) from (10) we use (10) repeatedly in applying it with $m=1$, then in $|A_1|$ with $m=2, \cdots$, and so forth, $k-1$ times and then with $m=n, n-1, \cdots, k+1$.

5. On the other hand, it is obviously not at all necessary or even advisable to use (10) in such a way as to obtain at each step one of the r_μ or l_μ, since *in any case* the number of the $\alpha_{\mu\nu}$ entering into the expressions corresponding to s_m is diminished by 1 at each step; we can further combine the use of the *columns* with that of the rows, as soon as the sums in the columns corresponding to s_m have become less than the moduli of the corresponding elements in the principal diagonal.

6. Our proof of the inequality (10) uses a very remarkable inequality valid for the elements of the inverse matrix of A, *when all σ_μ are less than* 1. Put

$$(12) \qquad A^{-1} = B = (b_{\mu\nu}), \qquad |b_{\mu\nu}| = \beta_{\mu\nu} \qquad (\mu, \nu = 1, \cdots, n).$$

Then we have

$$(13) \qquad \beta_{\mu\nu} \leqq \sigma_\mu \beta_{\nu\nu} \qquad (\mu \neq \nu),$$

and further

$$(14) \qquad b_{\mu\mu} = \frac{1}{a_{\mu\mu} + \theta_\mu t_\mu s_\mu}, \qquad |\theta_\mu| \leqq 1.$$

7. **Proof of (13).** We have for $\mu \neq \nu$

$$(15) \qquad \sum_\kappa a_{\mu\kappa} b_{\kappa\nu} = 0 \qquad (\mu \neq \nu).$$

Let B_ν be the maximum of $\beta_{\kappa\nu}$ in the νth column:

(16) $$\max_{\kappa} \beta_{\kappa\nu} = B_\nu.$$

Then we have from (15):

(17) $$\alpha_{\mu\mu}\beta_{\mu\nu} = | a_{\mu\mu}b_{\mu\nu} | = \left| -\sum_{\kappa\neq\mu} a_{\mu\kappa}b_{\kappa\nu} \right| \leqq B_\nu \sum_{\kappa\neq\mu} \alpha_{\mu\kappa} = \sigma_\mu\alpha_{\mu\mu}B_\nu,$$

and

(18) $$\beta_{\mu\nu} \leqq \sigma_\mu B_\nu < B_\nu \qquad (\mu \neq \nu).$$

But then it follows from (16) and (18) that $B_\nu = \beta_{\nu\nu}$, and (13) is proved.

8. **Proof of (14).** To prove (14) we start for a fixed μ from

(19) $$a_{\mu\mu}b_{\mu\mu} + \sum_{\kappa\neq\mu} a_{\mu\kappa}b_{\kappa\mu} = 1 \qquad (\mu = 1, \cdots, n)$$

and obtain by using (13) and (9):

(20) $$| a_{\mu\mu}b_{\mu\mu} - 1 | \leqq \sum_{\kappa\neq\mu} \alpha_{\mu\kappa}\beta_{\kappa\mu} \leqq \sum_{\kappa\neq\mu} \alpha_{\mu\kappa}\sigma_\kappa\beta_{\mu\mu} \leqq t_\mu\beta_{\mu\mu}s_\mu,$$

and therefore for a certain θ_μ with $|\theta_\mu| \leqq 1$:

(21) $$a_{\mu\mu}b_{\mu\mu} = 1 - \theta_\mu t_\mu s_\mu b_{\mu\mu}.$$

(14) follows now at once.

9. **Proof of (10′).** *We assume first that all σ_μ are less than* 1. Develop then A in the elements of the mth row; we obtain, denoting the algebraic complement of $a_{\mu\nu}$ by $A_{\mu\nu}$,

(22) $$A = A_m\left(a_{mm} + \sum_{\nu\neq m} a_{m\nu}\frac{A_{m\nu}}{A_{mm}}\right);$$

obviously $A_m = A_{mm}$ is not equal to 0, since Hadamard's theorem can be certainly applied to this principal minor of A.

On the other hand we have by (12) and (13):

(23) $$\frac{A_{m\nu}}{A_{mm}} = \frac{b_{\nu m}}{b_{mm}}, \qquad \left|\frac{A_{m\nu}}{A_{mm}}\right| = \frac{\beta_{\nu m}}{\beta_{mm}} \leqq \sigma_\nu \qquad (\nu\neq m),$$

and therefore

(24) $$\left|\frac{A}{A_m} - a_{mm}\right| \leqq \sum_{\nu\neq m} \alpha_{m\nu}\sigma_\nu \leqq t_m s_m.$$

(10′) is now proved, *if all σ_μ are less than* 1.

10. It is now easy to see that (10′) is also true if all σ_μ are less than or equal to 1. Indeed, multiply then all elements of A off the principal diagonal by t, $0<t<1$. We obtain a determinant for which (24) has been already proved and get thus with $t \uparrow 1$ the inequality (24) and thence (10′) under the assumption that all σ_μ are $\leqq 1$.

11. Suppose now that the hypothesis (11) is satisfied. We can then assume that in particular

$$\sigma_m > 1, \qquad t_m < 1. \tag{25}$$

Let then t be such that

$$t_m < t < \frac{1}{\sigma_m} < 1. \tag{26}$$

We multiply the mth row of A by t and divide the mth column by t. We obtain a new determinant A' and denote the expressions s_μ, σ_μ, and t_m corresponding to A' by s'_μ, σ'_μ, and t'_m. We have obviously

$$\sigma'_m = t\sigma_m < 1, \qquad s'_m = ts_m, \tag{27}$$

$$\sigma'_\mu \leqq \frac{1}{t}\sigma_\mu < 1 \qquad (\mu \neq m), \tag{28}$$

$$t'_m = \max_{\mu \neq m} \sigma'_\mu \leqq \frac{t_m}{t}\cdot \tag{29}$$

But then we can apply (10′) to A' and obtain by (27) and (29):

$$\frac{A}{A_m} - a_{mm} = \theta'_m t'_m s'_m = \theta_m t_m s_m, \tag{30}$$

where $|\theta_m| \leqq |\theta'_m| \leqq 1$. The proof of (10′) is now completed.

UNIVERSITY OF BASLE