2. ———, *The orthogonal development of non-linear functionals in series of Fourier-Hermite functionals*, Ann. of Math. vol. 48 (1947) pp. 385–392.

3. ———, *Non-linear integral equations*, Ann. of Math. vol. 51 (1950) pp. 629–642.

4. S. Kaczmarz and H. Steinhaus, *Theorie der Orthogonalreihen*, Warsaw-Lwow, 1935.

5. R. E. A. C. Paley, N. Wiener, and A. Zygmund, *Notes on random functions*, Math. Zeit. vol. 37 (1933) pp. 647–668.

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THE MULTIPLIER RULE FOR ORDINARY DIFFERENTIAL EQUATIONS

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One of the unsolved problems of the calculus of variations is to find a proof of the multiplier rule for the case of partial differential equations as side conditions. The essential difficulty lies in the fact that the theory of partial differential equations is not sufficiently developed to allow the use of the same proecedure which worked in the case of ordinary differential equations. Hence, it would be desirable to have a proof of the multiplier rule which made no appeal to the theory of differential equations, even in the case of ordinary differential equations. A proof of this type is the object of this paper.

Let us consider the problem of making stationary the functional $J[y]$ defined by

$$J[y] = \int_{x_0}^{x_1} F(x, y_1 \cdots y_n, y_1' \cdots y_n')dx \tag{1}$$

under boundary conditions which need not be specified here and under side conditions

$$G_j(x, y_1 \cdots y_n, y_1' \cdots y_n') = 0, \qquad j = 1, \cdots, p < n. \tag{2}$$

The usual proof of the multiplier rule starts from this problem and shows that the problem of making stationary the functional $K[y, \lambda]$,

$$K[y, \lambda] = \int_{x_0}^{x_1} \{F + \lambda_j G_j\} dx, \tag{3}$$

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under the same boundary conditions has the same solution, $u_1 \cdots u_n$, and that $K[y, \lambda]$ and $J[y]$ have the same stationary value.

Instead of following this approach, let us start with the second problem and try to show that it is equivalent to the first. Equating the first variation of $K[y, \lambda]$ with respect to the y_i to zero gives

$$a_{ij}\lambda_j' = b_{ij}\lambda_j + c_i, \qquad i = 1, \cdots, n; j = 1, \cdots, p, \tag{4}$$

where

$$a_{ij} = \frac{\partial G_j}{\partial y_i'}, \quad b_{ij} = \frac{\partial G_j}{\partial y_i} - \frac{d}{dx}\frac{\partial G_j}{\partial y_i'}, \quad c_i = \frac{\partial F}{\partial y_i} - \frac{d}{dx}\frac{\partial F}{\partial y_i'} \tag{5}$$

and the arguments of the functions are $x, u_1 \cdots u_n, u_1' \cdots u_n'$. Performing the same operation on $K[y, \lambda]$ with respect to the λ_j gives

$$G_j(x, u_1 \cdots u_n, u_1' \cdots u_n') = 0. \tag{6}$$

The equations (4) are n in number, but involve p functions λ_j. Hence, if the problem involving $K[y, \lambda]$ is to be meaningful, certain compatibility conditions must be satisfied. Let A be the determinant of the quantities a_{hj} $(h, j=1, \cdots, p)$ and let us make the inessential assumption $A \neq 0$. Then, the first p of the equations (4) can be put into diagonal form, namely

$$\lambda_j' = B_{hj}\lambda_h + C_j, \tag{7}$$

where the B_{hj} involve the a_{hj} and b_{hj}, and the C_j, the a_{hj} and c_j. Substituting (7) in the remaining $n-p$ equations of (4), we arrive at

$$(a_{kj}B_{hj} - b_{kh})\lambda_h = c_k - a_{kj}C_j, \qquad k = p+1, \cdots, n. \tag{8}$$

Let the rank of the equations (8) be r_1. It is clear that $r_1 \leqq \min(p, n-p)$. Then $n-p-r_1$ linear relations not involving the λ_h can be found among the expressions on the right-hand side of (8). These relations are necessary conditions on the functions $u_1 \cdots u_n$. Furthermore, r_1 of the functions λ_h can be expressed as linear combinations of the remaining $p-r_1$ λ_h. By means of these last relations, the equations (7) can be reduced to $p-r_1$ differential equations in the same number of variables and r_1 finite linear equations in $p-r_1$ variables.

We now have to deal with equations having the same form as (7) and (8). Hence the process indicated in the previous paragraph can be applied again to give r_1-r_2 necessary conditions on the $u_1 \cdots u_n$, r_2 of the λ_h can be expressed in terms of the remaining $p-r_1-r_2$ λ_h, and we shall be left with $p-r_1-r_2$ equations of the type of (7), and r_2

equations of the type of (8). For each step, this information can be put in the form of a line in a table.

Step	No. of Diff. Equations	No. of Finite Equations	Rank of Finite Equations	No. of Conditions on $u_1 \cdots u_n$	No. of Conditions on $\lambda_1 \cdots \lambda_p$
1	p	$n-p$	r_1	$n-p-r_1$	r_1
2	$p-r_1$	r_1	r_2	r_1-r_2	r_2
3	$p-r_1-r_2$	r_2	r_3	r_2-r_3	r_3
.					
.					
.					
m	$p-r_1- \cdots -r_{m-1}$	r_{m-1}	r_m	$r_{m-1}-r_m$	r_m
			Sum	$n-p-r_m$	$r_1+ \cdots +r_m$

This process can end in one of two ways, namely, either by the number of differential equations, or by the number of finite equations reducing to zero. If, at the $(m+1)$st step, the first alternative occurs, then a glance at the table shows that

$$p = r_1 + \cdots + r_m. \tag{9}$$

But then the table shows that we shall have exactly p independent relations among the λ_j, thus determining them uniquely as functions of $u_1 \cdots u_n$. By substituting those λ_j determined at the mth step in the differential equations which occurred at that step, we shall obtain $p-r_1- \cdots -r_{m-1}=r_m$ differential equations in the $u_1 \cdots u_n$. These, together with those already obtained, are $n-p$ in number. Combining the $n-p$ conditions and the equations (6), we shall have a system of n differential equations for the functions $u_1 \cdots u_n$.

In the event that the number of finite equations vanish at the $(m+1)$st step, it would follow that $r_m=0$. The table shows that we would have exactly $n-p$ conditions on the $u_1 \cdots u_n$ in this case. On the other hand, we would have $r_1+ \cdots +r_{m-1}$ finite equations in the λ_j and $p-r_1- \cdots -r_{m-1}$ differential equations. Hence, the λ_j would be determined by the $u_1 \cdots u_n$ only up to $p-r_1- \cdots -r_{m-1}$ arbitrary constants. These results, in contrast to the preceding case, do, however, depend on the theory of differential equations.

In either case, it is clear that if functions $u_1 \cdots u_n$ exist which satisfy the $n-p$ compatibility conditions, then also functions $\lambda_1 \cdots \lambda_p$ exist. If in addition, the $u_1 \cdots u_n$ satisfy the p equations (6), and the boundary conditions, the functional $K[y, \lambda]$ actually attains its stationary value. Consider the following remark due to R. Courant

(see reference). If a functional $J[u, v, \cdots]$ is stationary under certain side conditions for a given set of functions, and if this set of functions satisfies one or more relations, then J remains stationary for the same set of functions if one or more of these relations are added to the side conditions.

Now, by means of this remark, we can add the equations (6) to the side conditions of the problem. The functional $K[y, \lambda]$ turns into $J[y]$ and we are returned to the original problem. It is clear that the two problems have the same solution $u_1 \cdots u_n$, and the same stationary value. Since it has already been shown that the appropriate $\lambda_1 \cdots \lambda_p$ exist, both problems are meaningful. Hence, the multiplier rule is proved.

The author has already applied these ideas to the case of partial differential equations and has obtained some results. These will be published in a later paper.

Reference

R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, vol. 1, New York, Interscience, p. 201.

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