A UNIQUENESS THEOREM FOR A CLASS OF HARMONIC FUNCTIONS

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In this note we shall establish a uniqueness theorem for the class of functions \( u(r, \theta) \), harmonic in \( |z| < 1 \), for which

\[
\int_0^{2\pi} |u(r, \theta)| d\theta < M,
\]

where \( M \) is a finite constant independent of \( r \).

Theorem. Let \( u(r, \theta) \) be harmonic in \( |z| < 1 \) and there satisfy (1). Let \( \lim_{r \to 1} u(r, \theta) = 0 \) for almost all \( \theta \) in \( 0 \leq \theta \leq 2\pi \), and let \( \lim_{r \to 1} u(r, \theta) = \pm \infty \) for all \( \theta \) belonging to a countable set \( E \) in \( 0 \leq \theta \leq 2\pi \). If \( \lim_{r \to 1} u(r, \theta) \), wherever else it may exist, is not infinite, then there exist real constants \( c_n \) with \( \sum_{n=1}^{\infty} |c_n| < \infty \) such that, in \( |z| < 1 \),

\[
u(r, \theta) \equiv \sum_{n=1}^{\infty} c_n K(r, \theta - \theta_n),
\]

where \( \bigcup_{n=1}^{\infty} \theta_n = E \) and \( K(r, \theta - \alpha) = (1 - r^2)/(1 + r^2 - 2r \cos (\theta - \alpha)) \).

A harmonic function satisfying (1) in \( |z| < 1 \) has an integral representation

\[
u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} K(r, \theta - \phi) d\mu(\phi),
\]

where \( \mu(\phi) \) is of bounded variation in the interval \( 0 \leq \phi \leq 2\pi \). We consider the Lebesgue decomposition of \( \mu(\phi) \):

\[
u(\phi) = \nu'(\phi) + g(\phi),
\]

where \( \nu'(\phi) \) is absolutely continuous with \( \mu'(\phi) = \nu'(\phi) \) almost everywhere, and \( g(\phi) \) is of bounded variation with \( g'(\phi) = 0 \) almost everywhere. Now, for any \( \phi \) for which \( \mu'(\phi) \) exists, which is the case almost everywhere, \( \lim_{r \to 1} u(r, \phi) = \mu'(\phi) \). Since \( \lim_{r \to 1} u(r, \theta) = 0 \) almost everywhere, we have \( \nu'(\phi) = 0 \) almost everywhere, so that \( \nu(\phi) \) in (4) is identically constant. Now, it is known that, for any point of discontinuity \( \theta_0 \) of \( \mu(\phi) \), \( \lim_{r \to 1} u(r, \theta_0) = \pm \infty \). Consequently the points

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2 Cf. the author's paper The boundary values of a class of meromorphic functions, to appear in Duke Math. J.
of discontinuity of $\mu(\phi)$ are contained in the set $E$. If we write $g(\phi) = g_1(\phi) + \psi(\phi)$, where $g_1(\phi)$ is a continuous function of bounded variation with $g_1'(\phi) = 0$ almost everywhere and $\psi(\phi)$ is a step function, it follows from [1, pp. 127–128] that, unless $g_1(\phi)$ reduces to a constant, $g'(\phi)$ is infinite on a noncountable set of points. This implies that $\lim_{r\to1} u(r, \theta)$ is infinite on a noncountable set of values of $\theta$, which is contrary to hypothesis. Hence $g_1(\phi)$ is identically constant, and $\mu(\phi)$ reduces to a pure step function.

Since $\mu(\phi)$ in (3) reduces to a step function, we may replace the Stieltjes integral there by a series; more precisely: there exists a sequence of real numbers $\{c_n\}$ ($n = 1, 2, \cdots$) with $\sum_{n=1}^{\infty} |c_n| < \infty$ such that

$$u(r, \theta) = \sum_{n=1}^{\infty} c_n K(r, \theta - \theta_n),$$

and where $2\pi c_n$ represents the saltus of the step function $\psi(\phi)$ at the jump points $\theta_n$. Hence the theorem is proved.

The following corollary is an easy consequence of our theorem: Let $u(r, \theta)$ be harmonic in $|z| < 1$ and there satisfy (1); let $\lim_{r\to1} u(r, \theta) = 0$ for almost all $\theta$ in $0 \leq \theta \leq 2\pi$ and let $\lim_{r\to1} u(r, \theta)$, wherever else it may exist, not be infinite. Then $u(r, \theta)$ is identically zero in $|z| < 1$.

Indeed, since $\lim_{r\to1} u(r, \theta)$, wherever it exists, cannot be infinite, it follows that all the constants $c_n$ in (2) must be zero.

To show that condition (1) is essential we exhibit the function

$$u(r, \theta) = \frac{\partial}{\partial \theta} \left\{ \frac{1 - r^2}{1 + r^2 - 2r \cos \theta} \right\} = \frac{-2r(1 - r^2) \sin \theta}{[1 + r^2 - 2r \cos \theta]^2}$$

which is harmonic in $|z| < 1$ and has the property that, for all $\theta$, $\lim_{r\to1} u(r, \theta) = 0$.

**Bibliography**


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*We remark that P. C. Rosenbloom has proved a weaker form of this corollary assuming that $\lim_{r\to1} u(r, \theta) = 0$ for every value of $\theta$. His result will appear in his forthcoming book on partial differential equations.*

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