A THEOREM ON CONJUGATE NETS IN PROJECTIVE HYPERSPACE

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A theorem proved by C. C. Hsiung in a recent paper [1][1] may be stated as follows: *In a linear space* S_n *of* n $(\geqq 3)$ *dimensions let* N_x *be a conjugate net and* π *be a fixed hyperplane; then the points* M, $\overline{M}$ *of intersection of the fixed hyperplane* π *and the two tangents at a point* x *of the net* N_x *describe two conjugate nets* N_M, $N_{\overline{M}}$ *in the hyperplane* π, *respectively, and one of the two nets* N_M, $N_{\overline{M}}$ *is a Laplace transformed net of the other.* The purpose of this note is to prove, in an elementary manner, a general theorem of which the above stated theorem of Hsiung is a specialization. The statement of the theorem follows:

In a linear space S_n *of* n $(\geqq 3)$ *dimensions let* N_x *be a conjugate (parametric) net. Let* M, $\overline{M}$ *be points on the* u-, v-*tangents at* x *of the net* N_x, *respectively, which describe two nets* N_M, $N_{\overline{M}}$ *having the property that the tangent plane of* N_M $(N_{\overline{M}})$ *at* M $(\overline{M})$ *passes through* $\overline{M}$ (M). *The nets* N_M, $N_{\overline{M}}$ *are conjugate nets and each one of them is a Laplace transformed net of the other one.*

For the proof let us observe first that since N_x is a conjugate net, the points M, $\overline{M}$, $\partial\overline{M}/\partial u$, and $\partial M/\partial v$ lie in the tangent plane to N_x at x; this plane is therefore determined by the points M, $\partial\overline{M}/\partial u$, $\partial M/\partial v$. The conditions that the tangent planes to N_M and $N_{\overline{M}}$ at M and $\overline{M}$, respectively, pass through the points $\overline{M}$ and M are equivalent to the conditions that the matrices

$$\left(M,\ \frac{\partial M}{\partial u},\ \frac{\partial M}{\partial v},\ \overline{M}\right), \qquad \left(\overline{M},\ \frac{\partial \overline{M}}{\partial u},\ \frac{\partial \overline{M}}{\partial v},\ M\right)$$

be of rank three. It follows that $\partial\overline{M}/\partial u$ and $\partial M/\partial v$ are expressible by linear relations

$$\frac{\partial \overline{M}}{\partial u} = a\overline{M} + bM, \qquad \frac{\partial M}{\partial v} = \alpha M + \beta\overline{M}, \tag{1}$$

since the tangent planes to N_M and N_M at M and $\overline{M}$ cannot coincide with the tangent plane to N_x at a generic (nonplanar) point x of N_x. From the form of relations (1) it follows that each of the points M, $\overline{M}$ satisfies an equation of Laplace, and, therefore, each of the nets N_M, $N_{\overline{M}}$ is a conjugate net. Furthermore, the point $\overline{M}$ is the

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[1] Numbers in brackets refer to the references at the end of the note.

first Laplace transformed point of M with respect to the net N_M and the point M is the minus-first Laplace transformed point of $\overline{M}$ with respect to the net $N_{\overline{M}}$. These additional facts are easily established by use of equations (1) in verifying that $\overline{M}$ (M) is the point on the tangent to the v (u)-curve of N_M $(N_{\overline{M}})$ at M $(\overline{M})$ at which this line touches the edge of regression of the developable surface which it generates as the point M $(\overline{M})$ varies on the u (v)-curve of N_M $(N_{\overline{M}})$. For example, the point on the line $M\overline{M}$ where this line touches the edge of regression of the developable which it generates as M varies over the u-curve of N_M is the point $\sigma=\overline{M}+\mu M$ in which μ is determined so that the point $\partial\sigma/\partial u$ lies on the line $M\overline{M}$; that is to say, a linear relation

$$\frac{\partial\sigma}{\partial u}\equiv\frac{\partial\overline{M}}{\partial u}+\mu\frac{\partial M}{\partial u}+M\frac{\partial\mu}{\partial u}=cM+d\overline{M} \tag{2}$$

must be fulfilled. On substituting in (2) for $\partial\overline{M}/\partial u$ the right member of the first equation of (1) the resulting equation is

$$a\overline{M}+\left(b+\frac{\partial\mu}{\partial u}\right)M+\mu\frac{\partial M}{\partial u}=cM+d\overline{M}.$$

Since $\partial M/\partial u$ is linearly independent of M, $\overline{M}$ (the tangents to N_M being assumed distinct at M), μ must vanish. Hence the required point σ is found to be the point $\overline{M}$.

To show that the theorem of Hsiung is a specialization of this theorem the procedure is as follows. The intersections of the u- and v-tangents to N_x at x with the fixed hyperplane are the points M and $\overline{M}$, respectively. Since N_x is a conjugate net, the u-tangent at x generates a developable as x varies over the v-curve. Hence, the tangent line at M to the v-curve of N_M must lie in the tangent plane to N_x at x: that is to say, this tangent line is the line $M\overline{M}$ of intersection of the tangent plane to N_x at x and the hyperplane π. Similarly, the tangent at $\overline{M}$ to the u-curve of $N_{\overline{M}}$ passes through M. The conditions of the general theorem are therefore satisfied and the conclusion can be drawn.

Hsiung's theorem for $n=3$ is an analogue of a theorem of B. Su [2, p. 372] in which the conclusion is the same but the given net N_x is assumed to be an asymptotic net. If in the hypothesis of the general theorem proved in this note the net N_x is assumed to be an asymptotic net instead of a conjugate net, the same conclusion holds, and the theorem which results is a generalization of the theorem of Su. In the proof of this theorem, the details of which are omitted, the points $\partial M/\partial u$ and $\partial \overline{M}/\partial v$ are found to lie on the line $M\overline{M}$ because: (1) the osculating planes of the u- and v-curves of the net N_x coincide with the tangent plane to N_x at x, respectively, and (2) the tangent plane at M ($\overline{M}$) to the net N_M ($N_{\overline{M}}$) (assumed distinct from the tangent plane to N_x at x) passes through the point $\overline{M}$ (M).

References

1. C. C. Hsiung, *A general theory of conjugate nets in projective hyperspace*, Trans. Amer. Math. Soc. vol. 70 (1951) pp. 312–322.

2. B. Su, *On the surfaces whose Wilczynski quadrics all touch a fixed plane*, Revista de la Universidad Nacional de Tucumán (A) vol. 5 (1946) pp. 363–373.

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