Under these circumstances, it is clear that F is a blanket with domain

$$\sum_{f\in\mathfrak{K}} A_f.$$

We state the following theorem, the proof being omitted since it is somewhat lengthy, although straightforward.

7.2. THEOREM. *If F is the union of such a countable family of blankets $\mathfrak{K}$ that*

(i) *$f\in\mathfrak{K}$ implies f is ϕ-pseudo-strong;*

(ii) *the domains of the members of $\mathfrak{K}$ are disjointed;*

then F is ϕ-pseudo-strong.

This theorem insures that the blanket F enjoys the same differentiability properties as the members of $\mathfrak{K}$ themselves.

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ON THE SHEFFER A-TYPE OF POLYNOMIALS GENERATED BY $\phi(t)f(xt)$

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1. **Introduction.** Huff [1] treated polynomial sets $\{y_n(x)\}$ defined by a generating function,

$$\phi(t)f(xt) = \sum_{n=0}^{\infty} y_n(x)t^n, \tag{1}$$

in which

$$\phi(t) = \sum_{n=0}^{\infty} \frac{b_n t^n}{n!}; \qquad f(xt) = \sum_{n=0}^{\infty} \frac{a_n (xt)^n}{n!}, \tag{2}$$

with the restriction that neither b_0 nor any of the a_n can be zero. He proved, among other things, a theorem (his 2.1) which states a necessary and sufficient condition that the $\{y_n(x)\}$ above be of Sheffer A-type k. For definitions of A-type and a basic treatment of the important Sheffer classifications, see [2].

Starting with Huff's theorem, we shall obtain for $\{y_n(x)\}$ to be of

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Sheffer A-type k, a necessary and sufficient condition in the desirable form of a simple restriction upon the generating function in the definition of the polynomials.

2. **Form of the generating function.** Using the factorial function $(\alpha)_n$, so common in hypergeometric notation, we can restate Huff's theorem as follows:

THEOREM 1. *A set of polynomials* $\{y_n(x)\}$ *defined by* (1) *and* (2) *above is of Sheffer* A-*type k if, and only if, constants* $\gamma_0, \gamma_1, \cdots, \gamma_k$, *with* $\gamma_k \neq 0$, *exist such that, if q's are defined by*

$$a_s q_0 q_1 \cdots q_{s-1} = 1; \qquad s = 1, 2, 3, \cdots, \tag{3}$$

then $q_0 = \gamma_0$, *and*

$$q_s = \sum_{i=0}^{k} (-s)_i (-1)^i \gamma_i; \qquad s = 1, 2, \cdots, k. \tag{4}$$

Starting with Theorem 1 we shall prove the result:

THEOREM 2. *A necessary and sufficient condition that the set of polynomials* $\{y_n(x)\}$, *defined by* (1) *and* (2) *above, be of Sheffer* A-*type k is that the* $f(xt)$ *in* (1) *and* (2) *be a hypergeometric function with k denominator parameters and no numerator parameters, namely,*

$$f(xt) = {}_0F_k(-; \beta_1, \beta_2, \cdots, \beta_k; \sigma x t),$$

in which σ *is a nonzero constant.*

PROOF. First, assume that the set $\{y_n(x)\}$ is of Sheffer A-type k. Let the sequence of numbers $\gamma_0, \gamma_1, \cdots, \gamma_k$, with $\gamma_k \neq 0$, be that whose existence is guaranteed by Theorem 1. Define the numbers $\beta_1, \beta_2, \cdots, \beta_k$ by the identity in x,

$$\gamma_k \prod_{j=1}^{k} (\beta_j + x) \equiv \sum_{i=0}^{k} (-x)_i (-1)^i \gamma_i. \tag{5}$$

The q_s of Theorem 1 is given by (4) which, in view of (5), can be written

$$q_s = \gamma_k \prod_{j=1}^{k} (\beta_j + s); \qquad s = 0, 1, 2, \cdots. \tag{6}$$

From (6) we obtain by multiplication

$$\prod_{s=0}^{n-1} q_s = \gamma_k^n \prod_{j=1}^{k} (\beta_j)_n. \tag{7}$$

But, by (3),

$$a_n = \left[\prod_{s=0}^{n-1} q_s\right]^{-1},$$

so that

$$(8) \qquad a_n = \gamma_k^{-n}\left[\prod_{j=1}^{k} (\beta_j)_n\right]^{-1}.$$

From (8) and (2) it follows at once that

$$(9) \qquad f(xt) = {}_0F_k(-;\beta_1, \beta_2, \cdots, \beta_k; xt/\gamma_k).$$

Next suppose that the set of polynomials $\{y_n(x)\}$ is defined by a generating function of the stipulated kind,

$$(10) \qquad \phi(t)f(xt) = \phi(t){}_0F_k(-;\beta_1, \beta_2, \cdots, \beta_k; \sigma xt) = \sum_{n=0}^{\infty} y_n(x)t^n,$$

in which $\phi(t)$ is analytic and not zero at $t=0$. We need to show that the γ's of Theorem 1 exist.

The q's may be defined successively by (3) above; that is,

$$q_s = a_s/a_{s+1} = \sigma^{-1}\prod_{j=1}^{k} (\beta_j + s),$$

in terms of the given β's. The γ's are now defined by $\gamma_0 = q_0$ and

$$(11) \qquad q_s = \sum_{i=0}^{k} (-s)_i(-1)^i\gamma_i; \qquad s = 1, 2, \cdots, k.$$

Indeed, the explicit solution of equations (11) is

$$(12) \qquad \gamma_i = \sum_{s=0}^{i} \frac{(-1)^{i-s}q_s}{s!(i-s)!}; \qquad i = 0, 1, \cdots, k.$$

Equation (12) may be neatly expressed in symbolic notation by

$$(13) \qquad i!\gamma_i \doteq (q-1)^i.$$

That the γ_i of (12) satisfy (11) can be shown by direct substitution with rearrangement of finite series, here omitted.

The polynomials defined by (10) are given explicitly by

$$(14) \qquad y_n(x) = \sum_{i=0}^{n} \frac{b_{n-i}\sigma^i x^i}{(n-i)!i!(\beta_1)_i(\beta_2)_i \cdots (\beta_k)_i},$$

with b's given by (2). In the definition [2] of the Sheffer A-type of a set of polynomials, a dominant role is played by a differential operator J, defined in Sheffer's paper. The J for our $y_n(x)$ is the finite operator

$$J = \sum_{i=0}^{k} \gamma_i x^i D^{i+1}, \tag{15}$$

in which $D=d/dx$ and γ_i are those of equation (12). For the J of (15), $Jx^n = nq_{n-1}x^{n-1}$, from which it follows that $Jy_n(x) = y_{n-1}(x)$.

It will be useful in certain investigations now under way to have noted here that, if $\phi(t)$ is itself a hypergeometric function with p numerator parameters and r denominator parameters,

$$\phi(t) = {}_pF_r(\alpha_1, \alpha_2, \cdots, \alpha_p; \epsilon_1, \epsilon_2, \cdots, \epsilon_r; \rho t), \tag{16}$$

then the $y_n(x)$ are hypergeometric polynomials with $(r+1)$ numerator parameters and $(p+k)$ denominator parameters, namely,

$$y_n(x) = \psi_n F\begin{bmatrix} -n, 1-\epsilon_1-n, \cdots, 1-\epsilon_r-n; & \\ & (-1)^{p+r+1}\sigma x/\rho \\ 1-\alpha_1-n, \cdots, 1-\alpha_p-n, \beta_1, \cdots, \beta_k; & \end{bmatrix}, \tag{17}$$

with

$$\psi_n = \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n\rho^n}{n!(\epsilon_1)_n \cdots (\epsilon_r)_n}.$$

Equation (17) follows readily from (14) and the relation

$$(\alpha)_{n-i} = (-1)^i(\alpha)_n/(1-\alpha-n)_i.$$

References

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