MEAN CONVERGENCE OF ORTHOGONAL SERIES

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I. **Introduction.** Marcel Riesz's theorem [**9**, p. 153][1] about the mean convergence of Fourier series in L^p spaces has been extended by Harry Pollard to series of classical orthogonal polynomials [**2**; **3**; **4**; **5**]. In particular, it was shown that the Legendre expansion of a function $f \in L^p$ on $(-1, 1)$ converges in the mean of order p to f if $4/3 < p < 4$, but that this is false for any $p > 4$ (or $1 \leqq p < 4/3$). The question of the status of the limiting values $p = 4$ and $p = 4/3$ was raised but not settled by Pollard, and is repeated in the reviews of [**3**] and [**5**].

Entirely similar results were established for Jacobi series (for the limiting values of p, see the statements of Theorems 2 and 3 below), and for series of Bessel functions (Theorem 4), where the problem was also left open for $p = 4/3$ and $p = 4$ [**8**, pp. 797 and 802].

The purpose of the present paper is to show by a very simple argument that in all the above cases the property of mean convergence fails to hold at the limiting values of p.

II. **A necessary condition.** Suppose the sequence $\{\phi_n(x)\}$ is orthonormal with respect to a measure function $\mu(x)$ on (a, b), that is,

$$\int_a^b \phi_n(x)\phi_m(x)d\mu(x) = \delta_{mn}.$$

Let L_μ^p denote the space of functions f for which

$$\|f\|_{p,\mu} = \left\{\int_a^b |f(x)|^p d\mu(x)\right\}^{1/p} < \infty \qquad (p \geqq 1),$$

and put

$$a_n(f) = \int_a^b f(x)\phi_n(x)d\mu(x). \tag{1}$$

Then if the series

$$\sum_{n=0}^{\infty} a_n(f)\phi_n(x) \tag{2}$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

converges in L_μ^p for every $f \in L_\mu^p$, we must have

$$a_n(f)\|\phi_n\|_{p,\mu} = O(1) \tag{3}$$

for every $f \in L_\mu^p$. Hence the Banach-Steinhaus theorem [1, p. 80, Theorem 5] applied to the functionals (3) shows:

THEOREM 1. *If the series* (2) *converges in* L_μ^p *for every* $f \in L_\mu^p$, *then*

$$\|\phi_n\|_{p,\mu}\|\phi_n\|_{q,\mu} = O(1) \qquad \left(\frac{1}{p} + \frac{1}{q} = 1\right). \tag{4}$$

III. **Jacobi series and Bessel series.** Let $\alpha \geqq -1/2$, $\beta \geqq -1/2$, be fixed. The polynomials $P_n^{(\alpha,\beta)}(x)$ (we use Szegö's notation [6, p. 57]) are orthogonal on $(-1, 1)$ with respect to the weight function

$$w(x) = (1 - x)^\alpha(1 + x)^\beta.$$

If the sequence $\{c_n\}$ is determined by the conditions

$$\int_{-1}^{1} [c_n P_n^{(\alpha,\beta)}(x)]^2 w(x)dx = 1 \qquad (n = 0, 1, 2, \cdots; c_n > 0),$$

we put

$$\Phi_n(x) = c_n P_n^{(\alpha,\beta)}(x),$$

$$\Psi_n(x) = c_n P_n^{(\alpha,\beta)}(x)[w(x)]^{1/2}.$$

It is known that c_n is of order $n^{1/2}$ [5, formula (4.3.3)].

We suppose that $\alpha \geqq \beta$ (the results are completely analogous if $\alpha < \beta$), and put

$$r = \frac{4(\alpha + 1)}{2\alpha + 3}, \qquad s = \frac{4(\alpha + 1)}{2\alpha + 1}.$$

THEOREM 2. *If* $p = r$, *or if* $p = s$, *there exists a function* $f \in L_\mu^p$ *on* $(-1, 1)$ $(d\mu(x) = w(x)dx)$, *such that the series* $\sum a_n(f)\Phi_n(x)$ *does not converge in* L_μ^p.

Here

$$a_n(f) = \int_{-1}^{1} f(x)\Phi_n(x)w(x)dx.$$

THEOREM 3. *If* $p = 4/3$, *or if* $p = 4$, *there exists a function* $f \in L^p$ *on* $(-1, 1)$ *such that the series* $\sum a_n(f)\Psi_n(x)$ *does not converge in* L^p.

Here

$$a_n(f) = \int_{-1}^{1} f(x)\Psi_n(x)dx.$$

Next, let ξ be real, and $\nu \geqq -1/2$. Let $0<\lambda_1<\lambda_2< \cdots$ be the positive real zeros of the function $J_\nu(x) \cos \xi + xJ_\nu'(x) \sin \xi$. The functions $J_\nu(\lambda_n x)$ $(n=1, 2, 3, \cdots)$ are orthogonal on (0, 1) with respect to the weight function x. If the sequence $\{b_n\}$ is determined by the conditions

$$\int_0^1 [b_n J_\nu(\lambda_n x)]^2 x dx = 1 \qquad (n = 1, 2, 3, \cdots ; b_n > 0),$$

we put

$$B_n(x) = b_n J_\nu(\lambda_n x).$$

Let us note that the numbers λ_n are interlaced with the positive zeros of $J_\nu(x)$ [7, p. 480]. Since the latter are of order n [7, p. 506], so is λ_n. It follows that b_n is of order $n^{1/2}$ [7, p. 577].

THEOREM 4. *If $p=4/3$, or if $p=4$, there exists a function $f \in L_\mu^p$ on* (0, 1) *($d\mu(x)=xdx$) such that the series $\sum a_n(f)B_n(x)$ does not converge in L_μ^p.*

PROOFS. For Theorems 2 and 3 we use the formula [5, (8.21.18)]

$$P_n^{(\alpha,\beta)}(\cos \theta) = n^{-1/2}k(\theta)\{\cos (N\theta + \gamma) + (n \sin \theta)^{-1}O(1)\}, \tag{5}$$

where $N=n+(\alpha+\beta+1)/2$, $\gamma=-(2\alpha+1)\pi/4$, and $k(\theta)=\pi^{-1/2} \cdot (\sin (\theta/2))^{-\alpha-1/2}(\cos (\theta/2))^{-\beta-1/2}$.

Putting $x=\cos \theta$, and applying (5) on the interval $[n^{-1}, \pi-n^{-1}]$ on which the error term is uniform, a simple calculation shows that

$$\left\{\int_{-1}^{1} | \Phi_n(x) |^s w(x)dx\right\}^{1/s} > A_1(\log n)^{1/s}, \tag{6}$$

$$\left\{\int_{-1}^{1} | \Phi_n(x) |^r w(x)dx\right\}^{1/r} > A_2, \tag{7}$$

$$\left\{\int_{-1}^{1} \Psi_n^4(x)dx\right\}^{1/4} > A_3(\log n)^{1/4}, \tag{8}$$

$$\left\{\int_{-1}^{1} | \Psi_n(x) |^{4/3}dx\right\}^{3/4} > A_4. \tag{9}$$

Here $A_1, A_2, \cdots$ are positive constants. Theorems 2 and 3 fol-

low since (4) is violated in each case.

It should be noted that (6) was derived on the tacit assumption that $s<\infty$. If $s=\infty$, then $\alpha=-1/2$, $P_n^{(\alpha,\beta)}$ is a Tchebichef polynomial, and Theorem 2 reduces to a statement about trigonometric series which is well known [**9**, p. 155].

Finally, to prove Theorem 4, we put $y=\lambda_n x$ in the integrals involving $J_\nu(\lambda_n x)$, use the asymptotic formula [**7**, p. 199]

$$J_\nu(y) \sim \left(\frac{2}{\pi y}\right)^{1/2} \cos\left(y - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \qquad (y \to \infty),$$

and find that

$$\left\{\int_0^1 B_n^4(x)\,x\,dx\right\}^{1/4} > A_5(\log n)^{1/4}, \tag{10}$$

$$\left\{\int_0^1 |B_n(x)|^{4/3}\,x\,dx\right\}^{3/4} > A_6. \tag{11}$$

Thus (4) is again violated.

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