ON THE METHODS OF RAYLEIGH-RITZ-WEINSTEIN[1]

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The Rayleigh-Ritz-Weinstein methods are concerned with the connection between eigenvalues of a completely continuous symmetric operator in a Hilbert space and the eigenvalues of its projection into a subspace χ of H such that the difference $H\ominus\chi=\bar{\chi}$ is finite-dimensional.

In this paper we extend some of the known results of Weinstein and Aronszajn to more general symmetric operators and to the case where $\bar{\chi}$ is not necessarily finite-dimensional.

We consider a continuous symmetric operator Q and the solutions of the equation

$$(Q-\zeta I)x=0$$

for a given number ζ. Because of the continuity of Q, those solutions form a closed subspace H_ζ of H. If H_ζ is not empty, the number ζ is called an eigenvalue of Q and the elements of H_ζ the eigenvectors of Q corresponding to this eigenvalue. Because of the symmetry of Q the subspace H_ζ is empty for every nonreal ζ. Let us put $\overline{H}_\zeta=H\ominus H_\zeta$. We designate by P_ζ and $\overline{P}_\zeta$ the projectors associated with the subspaces H_ζ and $\overline{H}_\zeta$ respectively. Because of the symmetry of Q, the operator $Q-I\zeta$ transforms the whole space H into a subspace of $\overline{H}_\zeta$. *We shall assume that this subspace is identical with $\overline{H}_\zeta$, such that there is a self-adjoint operator R'_ζ which transforms H into $\overline{H}_\zeta$ and which satisfies the conditions*

$$(Q-I\zeta)R'_\zeta=R'_\zeta(Q-I\zeta)=\overline{P}_\zeta.$$

We assume further that R'_ζ is continuous. Let r be a positive number such that

$$\|R'_\zeta x\|<r\|x\|$$

holds for every x in H. The series.

$$\overline{P}_\zeta+(z-\zeta)R'_\zeta+(z-\zeta)^2R'^2_\zeta+\cdots+(z-\zeta)^nR'^n_\zeta+\cdots$$

then converges uniformly for $|z-\zeta|\leqq r$ and represents a continuous operator which satisfies the equation

$$(\overline{P}_\zeta-(z-\zeta)R'_\zeta)(\overline{P}_\zeta+(z-\zeta)R'_\zeta+\cdots+(z-\zeta)^nR'^n_\zeta+\cdots)=\overline{P}_\zeta.$$

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The operator $(Q-Iz)=(Q-I\zeta)-(z-\zeta)I$ then has an inverse R_z which is represented by

$$R_z = -\frac{P_\zeta}{z-\zeta} + \sum_{i=0}^{\infty} R_\zeta'^{\,i+1}(z-\zeta)^i \tag{1}$$

for every z which satisfies the inequality $0<|z-\zeta|\leq r$.

As an example of operators which satisfy the conditions stated above, without necessarily being completely continuous, we can consider the operator

$$Q = \sum_{i=1}^{\infty} k_i P_i$$

where $(k_1, k_2, \cdots, k_n, \cdots)$ is a bounded set of real numbers and $P_1, P_2, \cdots, P_n, \cdots$ a complete set of orthogonal projectors $(\sum_{i=1}^{\infty} P_i=I,\ P_iP_j=P_jP_i=0$ for $i\neq j)$. This operator obviously satisfies our conditions for every ζ which is not a limit point of the set $(k_1, k_2, \cdots, k_n, \cdots)$. In particular we have for this operator Q, $R_\zeta' = \sum_{k_i\neq\zeta} P_i/(k_i-\zeta)$.

1. **Weinstein's problem.** Let $\bar{\chi}$ be a given closed subspace of H, $\chi=H\ominus\bar{\chi}$. Find the solutions of the equation

$$P(Q-I\zeta)Px = 0 \tag{2}$$

where P is the projector associated to the subspace $\chi=PH$.

Since $P(Q-I\zeta)P$ is continuous, the set of all solutions of (2) is a closed subspace χ_ζ of χ.

We designate by $\bar{P}$ the projector $\bar{P}=I-P$ associated to the subspace $\bar{\chi}=\bar{P}H$ and consider the identity

$$\begin{aligned}(Q-I\zeta)x \equiv P(Q-I\zeta)Px &+ \bar{P}(Q-I\zeta)Px \\ &+ P(Q-I\zeta)\bar{P}x + \bar{P}(Q-I\zeta)\bar{P}x\end{aligned} \tag{3}$$

which connects the operator $Q-I\zeta$ with the operator $P(Q-I\zeta)P$. Hence x is an element of χ_ζ if and only if

$$(Q-I\zeta)x = \bar{P}(Q-I\zeta)x, \tag{4}$$

$$\bar{P}x = 0. \tag{5}$$

According to the equation (4), $x'=(Q-I\zeta)x$ is an element of

$$\bar{\chi}' = \bar{H}_\zeta \cap \bar{\chi}.$$

Let us put

$$\bar{\chi}'' = \bar{\chi} \ominus \bar{\chi}'.$$

We designate by $\overline{P}'$ and $\overline{P}''$ the projectors which correspond respectively to the subspaces $\bar{\chi}'$ and $\bar{\chi}''$. x' is then of the form $x'=\overline{P}'y$ where y is defined up to addition of an element of $\bar{\chi}''\oplus\chi$. Since x is an element of χ, we can choose this y in such a way that

$$P_{\mathfrak{r}}x = P_{\mathfrak{r}}y.$$

The elements x of $\chi_{\mathfrak{r}}$ are therefore of the form

$$x = R_{\mathfrak{r}}'\overline{P}'y + P_{\mathfrak{r}}y. \tag{6}$$

Conversely it is easy to verify that every element of the form (6) satisfies the equation (4). *The subspace $\chi_{\mathfrak{r}}$ is therefore the set of all those elements of the form* (6), *which satisfy the equation*

$$\overline{P}R_{\mathfrak{r}}'\overline{P}'y + \overline{P}P_{\mathfrak{r}}y = 0.$$

Let us consider first the elements of $\chi_{\mathfrak{r}}$ which are of the form $P_{\mathfrak{r}}y$. It is clear that those elements are in

$$\chi \cap H_{\mathfrak{r}} = H_{\mathfrak{r}}' = \chi'$$

and that every element of this subspace χ' is an element of the form $P_{\mathfrak{r}}y$ of $\chi_{\mathfrak{r}}$.

Remark. Since $H_{\mathfrak{r}}' = \chi'$ is the subset of all elements of $H_{\mathfrak{r}}$ orthogonal to $\bar{\chi}$, it is also the subset of all elements of $H_{\mathfrak{r}}$ orthogonal to

$$P_{\mathfrak{r}}\bar{\chi} \subseteq H_{\mathfrak{r}}.$$

We have therefore

$$P_{\mathfrak{r}}\bar{\chi} = H_{\mathfrak{r}} \ominus H_{\mathfrak{r}}' = H_{\mathfrak{r}}''.$$

On the other hand, since $\bar{\chi}'$ is the set of all elements of $\bar{\chi}$ orthogonal to $H_{\mathfrak{r}}$, we have

$$H_{\mathfrak{r}}'' = P_{\mathfrak{r}}\bar{\chi} = P_{\mathfrak{r}}\bar{\chi}'',$$

and

$$\bar{\chi}'' = \overline{P}H_{\mathfrak{r}} = \overline{P}H_{\mathfrak{r}}'',$$

and the correspondences

$$x \leftrightarrow y = P_{\mathfrak{r}}x, \qquad x \in \bar{\chi}'',\ y \in H_{\mathfrak{r}}'',$$
$$y \leftrightarrow x = \overline{P}y, \qquad x \in \bar{\chi}'',\ y \in H_{\mathfrak{r}}'',$$

are one-to-one correspondences. *The subspace $H_{\mathfrak{r}}$ is therefore the direct sum of two subspaces, of which one is the subspace $H_{\mathfrak{r}}' = \chi' = \chi \cap H_{\mathfrak{r}}$ and the other $H_{\mathfrak{r}}''$ which is isomorphic to $\bar{\chi}'' = \bar{\chi} \ominus (\overline{H}_{\mathfrak{r}} \cap \bar{\chi})$.*

Since

$$H = \bar{\chi} \oplus \chi = \bar{\chi}' \oplus \bar{\chi}'' \oplus \chi' \oplus \chi'' \quad \text{with} \quad \chi'' = \chi \ominus \chi'$$

we can replace, in the expression (6) of the element x of χ_ζ, the element y by the sum $y = y_1 + y_2 + y_3 + y_4$ of four orthogonal vectors $y_1 \in \bar{\chi}'$, $y_2 \in \bar{\chi}''$, $y_3 \in \chi''$, $y_4 \in \chi'$. In this way we obtain

$$x = R_\zeta' y_1 + P_\zeta(y_2 + y_3) + y_4.$$

The three terms of this expression are obviously orthogonal to each other. We have therefore

$$\chi_\zeta = \{ [R_\zeta' \bar{\chi}' \oplus P_\zeta(\bar{\chi}'' \oplus \chi'')] \cap \chi \} \oplus \chi'.$$

Since obviously

$$H_\zeta'' = P_\zeta \bar{\chi}'' = P_\zeta(\bar{\chi}'' \oplus \chi'') \qquad (P_\zeta \chi'' \subset H_\zeta'' = H_\zeta \ominus \chi'),$$

we can write the preceding relation as follows

$$\chi_\zeta = \{ [R_\zeta' \bar{\chi}' \oplus P_\zeta \bar{\chi}''] \cap \chi \} \oplus \chi'.$$

The subspace $\chi_\zeta \ominus \chi'$ is therefore the set of all elements of the form

$$R_\zeta' y + y', \qquad y \in \bar{\chi}', \; y' \in P_\zeta \bar{\chi}'' = H_\zeta'',$$

which satisfy the condition

$$\overline{P} R_\zeta' y + \overline{P} y' = 0. \tag{7}$$

Since $\overline{P} = \overline{P}' + \overline{P}''$ and since $\bar{\chi}'$ is orthogonal to H_ζ, this equation can be written as a system of two equations

$$\overline{P}' R_\zeta' \overline{P}' y = 0, \qquad y \in \bar{\chi}', \tag{8}$$

$$\overline{P} y' + \overline{P}'' R_\zeta' \overline{P}' y = 0, \qquad y' \in H_\zeta''. \tag{9}$$

We noticed before that the correspondence

$$y' \leftrightarrow \overline{P} y'$$

between $y' \in H_\zeta''$ and $\overline{P} y' \in \bar{\chi}'' = \overline{P} H_\zeta''$ is one-to-one. We can therefore define a continuous linear operator T such that

$$T \overline{P} y' = y', \qquad \text{for } y' \in H_\zeta'' \; (T \bar{\chi}'' = H_\zeta'' \subset H_\zeta).$$

So we obtain all solutions of (7) by putting

$$R_\zeta' y + y' = R_\zeta' y - T \overline{P}'' R_\zeta' y$$

where y is a solution of (8). *Let us designate by* $\bar{\chi}_1'$ *the subspace of all elements* y *of* $\bar{\chi}'$ *which satisfy* (8). *Then we can write*

$$\chi_\zeta \ominus \chi' = (R_\zeta' - T\overline{P}''R_\zeta')\bar{\chi}_1'.$$

We remark further that *the correspondence*

$$y \leftrightarrow (R_\zeta' - T\overline{P}''R_\zeta')y$$

between elements of $\bar{\chi}_1'$ *and* $\chi_\zeta \ominus \bar{\chi}'$ *is one-to-one*, since from

$$(R_\zeta' - T\overline{P}''R_\zeta')y = 0$$

follows

$$y = (Q - I\zeta)(R_\zeta' - T\overline{P}''R_\zeta')y = 0.$$

Now we are going to express the result of our analysis in terms of R_z, which is an analytic function of z for $0 < |z-\zeta| \leq r$. From the representation

$$R_z = -\frac{P_\zeta}{z - \zeta} + \sum_{i=0}^{\infty} R_\zeta'^{\,i+1}(z - \zeta)^i$$

of this operator, one sees that the space $\bar{\chi}(z)$ of the functions

$$y(z) = \overline{P}R_z\overline{P}y \qquad \text{with} \quad y \in \bar{\chi} = \bar{\chi}'' \oplus \bar{\chi}_1' \oplus \bar{\chi}_0'$$

is the direct sum of three subspaces which are respectively isomorphic to $\bar{\chi}''$, $\bar{\chi}_0' = \bar{\chi}' \ominus \bar{\chi}_1'$, $\bar{\chi}_1'$:

I. The set of all functions of the form

$$\overline{P}R_z\overline{P}y = -\frac{\overline{P}P_\zeta\overline{P}y}{z - \zeta} + \sum_{i=0}^{\infty} \overline{P}R_\zeta'^{\,i+1}\overline{P}(z - \zeta)^i y, \qquad y \in \bar{\chi}'', \tag{10}$$

which are completely characterized by their principal parts, since the operator $\overline{P}P_\zeta\overline{P}$ transforms $\bar{\chi}''$ into itself in a one-to-one manner.

II. The set of all functions of the form

$$\overline{P}R_z\overline{P}y = \sum_{i=0}^{\infty} \overline{P}R_\zeta'^{\,i+1}\overline{P}(z - \zeta)^i y, \qquad y \in \bar{\chi}_0', \tag{11}$$

which are completely characterized by their first terms

$$\overline{P}R_\zeta'\overline{P}y,$$

since $\bar{\chi}_1'$ contains all elements y of $\bar{\chi}'$ for which $\overline{P}'R_\zeta'\overline{P}'y$ is zero.

III. The set of all functions of the form

$$\overline{P}R_z\overline{P}y = \sum_{i=1}^{\infty} \overline{P}(R_\zeta')^{i+1}\overline{P}(z - \zeta)^i y, \qquad y \in \bar{\chi}_1', \tag{12}$$

which are also characterized by their first terms

$$(z-\zeta)\bar{P}R_\zeta'^2\bar{P}y,$$

since, as we shall show right now, $\bar{P}R_\zeta'^2\bar{P}y=0, y\in\bar{\chi}_1'$ if and only if $y=0$.

We are going to show now that *the expression*

(13) $\bar{P}P_\zeta\bar{P}y_1+\bar{P}R_\zeta'\bar{P}y_2+\bar{P}R_\zeta'^2\bar{P}y_3$, with $y_1\in\bar{\chi}'', y_2\in\bar{\chi}_0', y_3\in\bar{\chi}_1'$,

vanishes if and only if $y_1=y_2=y_3=0$. For Im $\zeta\neq 0$, the subspaces H_ζ and χ_ζ are empty, since Q and PQP are self-adjoint operators. The subspaces $\bar{\chi}''$ and $\bar{\chi}_1'$ of $\bar{\chi}$ are then empty[2] and $\bar{\chi}=\bar{\chi}_0$ so that our proposition becomes obvious for Im $\zeta\neq 0$. Let us assume now that ζ is real. Since y_2 and y_3 are elements of $\bar{\chi}'=\bar{\chi}\cap\bar{H}_\zeta$ and since $\bar{P}P_\zeta\bar{P}y_1\in\bar{\chi}''$, the expression (13) vanishes only if

$$\bar{P}'R_\zeta'\bar{P}'y_2+\bar{P}'R_\zeta'^2\bar{P}'y_3=0.$$

From this equality we can deduce

$$(y_3,\bar{P}'R_\zeta'\bar{P}'y_2)+(y_3,\bar{P}'R_\zeta'^2\bar{P}'y_3)=0.$$

But since

$$(y_3,\bar{P}'R_\zeta'\bar{P}'y_2)=(\bar{P}'R_\zeta'\bar{P}'y_3,y_2)=(0,y_2)=0,$$

and

$$(y_3,\bar{P}'R_\zeta'^2\bar{P}'y_3)=(R_\zeta'y_3,R_\zeta'y_3),$$

we must have $R_\zeta'y_3=0$ which would imply $y_3=\bar{P}_\zeta y_3=(Q-I_\zeta)R_\zeta'y_3=0$. On the other hand, $\bar{P}'R_\zeta'\bar{P}'y_2$ ($y_2\in\bar{\chi}_0'$) vanishes if and only if $y_2=0$. The vanishing of (13) would therefore imply $y_2=y_3=0$, and $\bar{P}P_\zeta\bar{P}y_1=0$ which implies $y_1=0$.

We can now formulate the result of our analysis as follows:

THEOREM. *The subspaces $H_\zeta\ominus(H_\zeta\cap\chi)$ and $\chi_\zeta\ominus(H_\zeta\cap\chi)$ are respectively equal to*

$$P_\zeta\bar{\chi}''\quad\textit{and}\quad(R_\zeta'-T\bar{P}''R_\zeta')\bar{\chi}_1'$$

with $\bar{\chi}''\oplus\bar{\chi}_0'\oplus\bar{\chi}_1'=\bar{\chi}$, where $\bar{\chi}'=\bar{\chi}_0'\oplus\bar{\chi}_1'$ is the subspace of $\bar{\chi}$ for which $\bar{P}R_z\bar{P}$ is regular at the point $z=\zeta$ and $\bar{\chi}_1'$ the subspace for which $\bar{P}R_z\bar{P}$ has a zero at $z=\zeta$. The correspondences defined by the operators P_ζ, $R_\zeta'-T\bar{P}''R_\zeta'$ between

$$\bar{\chi}''\quad\textit{and}\quad P_\zeta\bar{\chi}'',$$
$$\bar{\chi}_1'\quad\textit{and}\quad(R_\zeta'-T\bar{P}''R_\zeta')\bar{\chi}_1',$$

respectively, are one-to-one correspondences. The subspaces of the co-

[2] Since $\bar{\chi}''$ and $\bar{\chi}_1'$, are respectively isomorphic to $H_\zeta\ominus(H_\zeta\cap\chi)$ and $\chi_\zeta\ominus(H_\zeta\cap\chi)$.

efficients of the first terms of the functions

$$y_1(z) = \bar{P}R_z\bar{P}y_1, \qquad y_1 \in \bar{\chi}'',$$
$$y_2(z) = \bar{P}R_z\bar{P}y_2, \qquad y_2 \in \bar{\chi}_0',$$
$$y_3(z) = \bar{P}R_z\bar{P}y_3, \qquad y_3 \in \bar{\chi}_1',$$

are the three linearly independent subspaces

$$\bar{P}P_\zeta\bar{P}\bar{\chi}'', \qquad \bar{P}R_\zeta'\bar{P}\bar{\chi}_0', \qquad \bar{P}R_\zeta'^2\bar{P}\bar{\chi}_1'$$

of $\bar{\chi}$ which are respectively isomorphic to $\bar{\chi}''$, $\bar{\chi}_0'$, $\bar{\chi}_1'$.

This statement includes the main known property of Weinstein's determinant. In fact let us consider the special case where the subspace $\bar{\chi}$ is finite-dimensional. Then

$$H_\zeta \ominus (H_\zeta \cap \chi) = P_\zeta\bar{\chi}'', \qquad \chi_\zeta \ominus (H_\zeta \cap \chi) = (R_\zeta' - T\bar{P}''R_\zeta')\bar{\chi}_1',$$
$$\bar{\chi}'', \qquad \bar{\chi}_0', \qquad \bar{\chi}_1',$$
$$\bar{P}P_\zeta\bar{P}\bar{\chi}'', \qquad \bar{P}R_\zeta'\bar{P}\bar{\chi}_0', \qquad \bar{P}R_\zeta'^2\bar{P}\bar{\chi}_1'$$

are finite too and our theorem states that

$$\dim (H_\zeta \ominus (H_\zeta \cap \chi)) = \dim \bar{\chi}'',$$
$$\dim (\chi_\zeta \ominus (H_\zeta \cap \chi)) = \dim \bar{\chi}_1',$$
$$\bar{\chi} = \bar{\chi}'' \oplus \bar{\chi}_0' \oplus \bar{\chi}_1' = \bar{P}P_\zeta\bar{P}\bar{\chi}'' + \bar{P}P_\zeta'\bar{P}\bar{\chi}_0' + \bar{P}R_\zeta'^2\bar{P}\bar{\chi}_1'$$

with

$$\nu'' = \dim \bar{\chi}'' = \dim \bar{P}P_\zeta\bar{P}\bar{\chi}'',$$
$$\nu_0' = \dim \bar{\chi}_0' = \dim \bar{P}R_\zeta'\bar{P}\bar{\chi}_0',$$
$$\nu_1' = \dim \bar{\chi}_1' = \dim \bar{P}R_\zeta'^2\bar{P}\bar{\chi}_1'.$$

Let $e_1'', e_2'', \cdots, e_{\nu''}''; e_1'(0), e_2'(0), \cdots, e_{\nu_0'}'(0); e_1'(1), e_2'(1), \cdots, e_{\nu_1'}'(1)$ be a system of orthonormal bases of $\bar{\chi}''$, $\bar{\chi}_0'$, $\bar{\chi}_1'$ respectively. The Weinstein determinant $|\bar{P}R_z\bar{P}|$ will have the following expression:

$$|\bar{P}R_z\bar{P}| \equiv \begin{vmatrix} \bar{P}R_z\bar{P}e_i'' \\ \bar{P}R_z\bar{P}e_i'(0) \\ \bar{P}R_z\bar{P}e_i'(1) \end{vmatrix}$$
$$= (z-\zeta)^{\nu_1'-\nu''}\left\{\begin{vmatrix} -\bar{P}P_\zeta\bar{P}e_i'' \\ \bar{P}R_\zeta'\bar{P}e_i'(0) \\ \bar{P}R_\zeta'^2\bar{P}e_i'(1) \end{vmatrix} + (z-\zeta)A_1 + \cdots\right\}$$

where

$$\begin{vmatrix} -\overline{P}P_\zeta\overline{P}e_i' \\ \overline{P}R_\zeta'\overline{P}e_i'(0) \\ \overline{P}R_\zeta'^2\overline{P}e_i'(1) \end{vmatrix} \neq 0,$$

since the vectors $\overline{P}P_\zeta\overline{P}e_1''$, $\cdots$, $\overline{P}P_\zeta\overline{P}e_{\nu''}''$; $\overline{P}R_\zeta'\overline{P}e_1'(0)$, $\cdots$, $\overline{P}R_\zeta'\overline{P}e_{\nu_0'}'(0)$; $\overline{P}R_\zeta'^2\overline{P}e_1'(1)$, $\cdots$, $\overline{P}R_\zeta'^2\overline{P}e_{\nu_1'}'(1)$ are linearly independent. The order $\nu_\zeta(|\overline{P}R_z\overline{P}|)$ of the function $|\overline{P}R_z\overline{P}|$ at the point $z=\zeta$ is therefore

$$\begin{aligned}\nu_\zeta(|\overline{P}R_z\overline{P}|) &= \dim \bar{\chi}_1' - \dim \bar{\chi}'' \\ &= \dim (\chi_\zeta \ominus (\chi \cap H_\zeta)) - \dim (H_\zeta \ominus (\chi \cap H_\zeta)).\end{aligned}$$

This is the main property of Weinstein's determinant as stated by Aronszajn.

2. **Aronszajn's problem.** Let us consider now the following problem which is, in a certain way, the inverse of the preceding:

We assume that we know the resolvent

$$\rho_z = -\frac{\pi_\zeta}{z-\zeta} + \sum_{i=0}^{\infty} \rho_\zeta'^{\,i+1}(z-\zeta)^i$$

of $(PQP-Iz)$ in the neighborhood of the point $z=\zeta$ in the subspace χ and we want to determine the subspace H_ζ of all solutions x of

$$(Q - I\zeta)x = 0 \tag{14}$$

in H. Here π_ζ represents the projector associated with χ_ζ, that is, $\chi_\zeta=\pi_\zeta H$. We shall designate by $\bar{\pi}_\zeta$ the projector associated with $\bar{\chi}_\zeta=\chi\ominus\chi_\zeta$ and we shall have

$$\begin{aligned}\rho_z(PQP - Pz) &= (PQP - Pz)\rho_z = P,\\ \rho_\zeta'(PQP - P\zeta) &= (PQP - P\zeta)\rho_\zeta' = \bar{\pi}_\zeta.\end{aligned}$$

We consider again the identity (3). The first two terms of the left side of this identity are vectors of the subspace χ while the last two terms are vectors of the subspace $\bar{\chi}$. $(Q-I\zeta)x$ will therefore vanish if and only if

$$P(Q - I\zeta)Px + P(Q - I\zeta)\overline{P}x = 0, \tag{15}$$

$$\overline{P}(Q - I\zeta)Px + \overline{P}(Q - I\zeta)\overline{P}x = 0. \tag{16}$$

Thus the solutions of (14) are furnished by

$$x = Px + \overline{P}x$$

where $\{Px, \bar{P}x\}$ is a solution of the system of two linear equations (15), (16) with the unknowns Px and $\bar{P}x$. Let us assume that ζ is not an eigenvalue of PQP. The projector π_ζ is then zero and $P(Q-I\zeta)P$ has a known inverse

$$\rho_\zeta = \rho_\zeta'$$

in χ. We can solve in this case the equation (15) with respect to Px and substitute in (16). In this way we get

$$Px = -\rho_\zeta P(Q - I\zeta)\bar{P}x,$$
$$-\bar{P}(Q - I\zeta)\rho_\zeta P(Q - I\zeta)\bar{P}x + \bar{P}(Q - I\zeta)\bar{P}x = 0.$$

The function

$$\begin{aligned} &-\bar{P}(Q - Iz)\rho_z P(Q - Iz)\bar{P}x + \bar{P}(Q - Iz)\bar{P}x \\ &= \bar{P}(I - (Q - Iz)\rho_z P)(Q - Iz)\bar{P}x = \bar{P}(Q - Iz)(I - \rho_z P(Q - Iz))\bar{P}x \end{aligned}$$

has therefore a zero at $z=\zeta$ for $\bar{P}x \in \bar{P}H_\zeta = \bar{\chi}''$. But every element of $\bar{\chi}''$ is the coefficient of the principal part of a function of the form $\bar{P}R_z\bar{P}x$ which has a simple pole of first order at the point $z=\zeta$ so that we can expect that the product

$$\bar{P}(I - (Q - Iz)\rho_z P)(Q - Iz)\bar{P}\cdot\bar{P}R_z\bar{P}$$

has no pole at all at the point $z=\zeta$. In fact this product can be computed as follows:

$$\begin{aligned} &\bar{P}(I - (Q - Iz)\rho_z P)(Q - Iz)\bar{P}R_z\bar{P} \\ &\qquad = \bar{P}(I - (Q - Iz)\rho_z P)(Q - Iz)(I - PR_z)\bar{P} \\ &\qquad = \bar{P}(I - (Q - Iz)\rho_z P)(Q - Iz)(R_z - PR_z)\bar{P} \\ &\qquad = \bar{P}(I - (Q - Iz)\rho_z P)(I - (Q - Iz)PR_z)\bar{P} \\ &\qquad = \bar{P}(I - (Q - Iz)PR_z - (Q - Iz)\rho_z P + (Q - Iz)PR_z)\bar{P} = \bar{P}. \end{aligned}$$

One can verify in the same way that

$$\bar{P}R_z\bar{P}\cdot\bar{P}(Q - Iz)(I - \rho_z P(Q - Iz))\bar{P} = \bar{P}.$$

The operators $\bar{P}R_z\bar{P}$ *and*

$$-\bar{P}\cdot(Q - Iz)\rho_z P(Q - Iz)\bar{P} + \bar{P}(Q - Iz)\bar{P} = \bar{P}(Q - Iz - Q\rho_z PQ)\bar{P}$$

are therefore the inverse of each other in $\bar{\chi}$.

This proposition contains the known Aronszajn result that in the case where $\bar{\chi}$ is finite, the determinant $|\bar{P}(Q-Iz+Q\rho_z Q)\bar{P}|$ is the inverse of the Weinstein determinant $|\bar{P}R_z\bar{P}|$.

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ON SOME FUNCTIONS HOLOMORPHIC IN AN INFINITE REGION

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S. Mandelbrojt indicated the following proposition: If a function is holomorphic and bounded in a half-strip of the z-plane containing the half-axis ox as a part of its central line and if this function and a certain infinite sequence of its derivatives vanish at the origin, then it is identically zero. The proof of this proposition is based upon a result of Mandelbrojt [1, p. 372].[2] In the present paper, we consider a function $F(z)$ holomorphic in a region Δ of the z-plane defined by $x \geqq d$, $|y| \leqq g(x)$, where $-\infty < d < 0$ and where $g(x)$ is a certain positive continuous function tending to zero with $1/x$. In this case if, in Δ, $F(z)$ tends to zero rapidly enough and uniformly with respect to y as x tends to infinity, and if $F(z)$ and a certain infinite sequence of its derivatives vanish at the origin, then $F(z)$ is identically zero. In order to establish our proposition, we prove at first a lemma by means of the following theorem of G. Valiron [3, p. 62, §32]:

Theorem V. *Let $Y(X)$ be a real function having a first derivative for $X \geqq X_0$ such that*

$$\lim_{x=\infty} \frac{XY'(X)}{\psi(X)} = 1; \qquad \psi(X) \geqq 1, \quad X \geqq X_0; \qquad \lim_{x=\infty} \frac{X\psi'(X)}{[\psi(X)]^2} = 0.$$

Let $\Phi(X)$ be an entire function and let $M(r) = \max_{|z|=r} |\Phi(z)|$. Then a necessary and sufficient condition that

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[2] Numbers in brackets refer to the bibliography at the end of this paper.