AN EXTENSION OF RENEWAL THEORY

KAI LAI CHUNG[1] AND HARRY POLLARD

Renewal theory can be and has been viewed in several different ways. One way is to reduce the problems to those concerning the addition of independent, non-negative random variables having a common distribution. Accordingly we introduce the random variables $X_1, X_2, \cdots$, independent, non-negative, and all possessing the same distribution function $F(x)$ with mean $m=\int_{-\infty}^{\infty} x dF(x)$ $(0<m \leqq \infty)$, and their successive sums $S_n = \sum_{k=1}^{n} X_k$. Either $F(x)$ is purely discontinuous with all its discontinuities located at the multiples of a fixed real number or it is not so; we call the first case the lattice case and the second the nonlattice case. In the lattice case there is no real loss of generality by assuming (as we shall do in the following) the said discontinuities are all located at integers whose greatest common divisor is one. One of the main results of renewal theory can then be stated as follows:

(i) *In the lattice case, if x runs through integers*:

$$\lim_{x\to\infty} \sum_{n=1}^{\infty} P(S_n = x) = \frac{1}{m} \tag{1}$$

(Erdös, Feller, and Pollard [1]; an equivalent theorem had been given by Kolmogorov [2] previously).

(ii) *In the nonlattice case, if h is any positive number*:

$$\lim_{x\to\infty} \sum_{n=1}^{\infty} P(x \leqq S_n \leqq x + h) = \frac{h}{m} \tag{2}$$

(Doob [3], Blackwell [4]). In both cases $1/m=0$ if $m=\infty$.

In renewal theory only non-negative random variables are considered but the formulas (1) and (2) remain meaningful if we drop the assumption of non-negativeness. In other words, it is legitimate to inquire whether they are still true when the random variables are allowed to assume both positive and negative values, but otherwise subject to the same conditions as before. Our answer to this question is affirmative, but not without some restrictions which we believe to be unnecessary but which we are unable to remove at present.[2] To be precise, we can prove the following:

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[2] Added in proof. In the meantime T. E. Harris and D. Blackwell informed us that they had removed these restrictions. Their methods are entirely different from ours.

(i) *In the lattice case, if x runs through integers and $m \neq \pm\infty$, then*

$$\lim_{x\to\infty} \sum_{n=1}^{\infty} P(S_n = x) = \begin{cases} 1/m & \text{if } m > 0, \\ 0 & \text{if } m < 0. \end{cases} \tag{3}$$

(ii) *In the nonlattice case, if h is any positive number and if we assume in addition that* $\limsup_{t\to\infty} |f(t)| < 1$ *where* $f(t) = \int_{-\infty}^{\infty} e^{itx} dF(x)$, *and $m \neq \pm\infty$, then*

$$\lim_{x\to\infty} \sum_{n=1}^{\infty} P(x \leqq S_n \leqq x + h) = \begin{cases} h/m & \text{if } m > 0, \\ 0 & \text{if } m < 0. \end{cases} \tag{4}$$

Analogies for $x \to -\infty$ are obvious. The additional assumption made in (ii) is satisfied if for example $F(x)$ contains a nonvanishing absolutely continuous part. (This was Doob's original assumption about (2) later removed by Blackwell.) It should be mentioned that if $m = 0$, then both series in (3) and (4) diverge for every integral and real x, respectively; see [5].

In the following we shall prove only (4), for two reasons. First, the proof of (3) is entirely similar and indeed simpler; second, another simpler proof of (3) has subsequently been found by Chung and Wolfowitz which will appear elsewhere [6].

One more remark: The method we use, that of Fourier inversion and direct calculation, seems new in this connection. It is suggested by a similar approach to another, not unrelated, problem employed by Chung and Fuchs [5]. The actual calculation, however, turns out to be quite different.

1°. To prove (4) it is sufficient to prove that

$$\lim_{x\to\infty} \lim_{r\to 1-0} \sum_{n=1}^{\infty} r^n \int_0^h P(|S_n - x| \leqq u) du = \begin{cases} h^2/m & \text{if } m > 0, \\ 0 & \text{if } m < 0. \end{cases} \tag{5}$$

For, the existence of the inner limit implies that

$$L(x, u) = \sum_{n=1}^{\infty} P(|S_n - x| \leqq u) < \infty$$

for all x and u, and moreover that

$$\lim_{x\to\infty} \int_0^h L(x, u) du = \begin{cases} h^2/m & \text{if } m > 0, \\ 0 & \text{if } m < 0. \end{cases} \tag{6}$$

Suppose $m > 0$. If $0 < h_1 < h_2$, since $L(x, u)$ is nondecreasing in u,

$$L(x, h_1)(h_2 - h_1) \leqq \int_{h_1}^{h_2} L(x, u) du \leqq L(x, h_2)(h_2 - h_1).$$

Letting $x \to \infty$ we obtain, using (6),

$$\limsup_{x\to\infty} L(x, h_1) \leqq \frac{h_1 + h_2}{m} \geqq \liminf_{x\to\infty} L(x, h_2).$$

Letting $h_2 \downarrow h_1$ in the first and $h_1 \uparrow h_2$ in the second inequality we obtain

$$\limsup_{x\to\infty} L(x, h_1) \leqq \frac{2h_1}{m}, \qquad \liminf_{x\to\infty} L(x, h_2) \geqq \frac{2h_2}{m}.$$

Hence $\lim_{x\to\infty} L(x, h) = 2h/m$ which is equivalent to (4). The case $m < 0$ is even simpler.

2°. Using a well known Fourier inversion formula we can write the sum in (5) as

$$\begin{aligned} &\sum_{n=0}^{\infty} r^n \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-itx}\frac{1-\cos ht}{t^2}(f(t))^n dt \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos ht}{t^2}\,\frac{e^{-itx}}{1-rf(t)}\,dt \end{aligned} \tag{7}$$

if $0<r<1$ and $x>h$, the term corresponding to $n=0$ vanishing. We wish to evaluate

$$\lim_{x\to+\infty}\ \lim_{r\to 1-0}\ \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1-\cos ht}{t^2}\,\frac{e^{-itx}}{1-rf(t)}\,dt. \tag{8}$$

3°. To do this we first show that for any $\delta > 0$

$$\lim_{x\to\infty}\ \lim_{r\to 1-0}\int_{|t|\geqq\delta}\frac{1-\cos ht}{t^2}\,\frac{e^{-itx}}{1-rf(t)}\,dt = 0. \tag{9}$$

Since $\limsup_{t\to\infty}|f(t)|<1$ we have $\sup_{|t|\geqq\delta}|f(t)| = \epsilon(\delta) < 1$. Thus we can take the limit with respect to r inside the integral sign in (9). (9) then follows by the Riemann-Lebesgue lemma in Fourier analysis.

Let $\epsilon = (1-r)/r$, then $1 - rf(t) = r\{\epsilon + [1-f(t)]\}$. Taking the real part of (8), as we plainly may, and confining ourselves with the range $|t| < \delta$, on account of (9), we must now show that

$$\begin{aligned} &\lim_{\delta\to 0}\ \lim_{x\to\infty}\ \lim_{\epsilon\to 0}\ \frac{1}{\pi}\int_{|t|<\delta}\frac{1-\cos ht}{t^2}\,\Re\left\{\frac{e^{-itx}}{\epsilon + 1 - f(t)}\right\}dt \\ &= \frac{h^2}{2}\left(\frac{1}{m} + \frac{1}{|m|}\right). \end{aligned} \tag{10}$$

The extra limit in δ is added for the sake of convenience.

Now we have

$$1 + imt - f(t) = \int_{-\infty}^{\infty} (1 - \cos tx) dF(x) + i \int_{-\infty}^{\infty} (tx - \sin tx) dF(x).$$

Let

$$R = R(t) = \int_{-\infty}^{\infty} (1 - \cos tx) dF(x),$$

$$I = I(t) = \int_{-\infty}^{\infty} (tx - \sin tx) dF(x).$$

Then $R(t) \geqq 0$, $R(t)$ is even and $I(t)$ is odd. We have

$$\mathfrak{R}\left\{\frac{e^{-itx}}{\epsilon + 1 - f(t)}\right\} = \frac{(\epsilon + R) \cos tx + (mt - I) \sin tx}{(\epsilon + R)^2 + (mt - I)^2}.$$

4°. Consider

$$\lim_{x\to\infty} \lim_{\epsilon\to\infty} \frac{1}{\pi} \int_{|t|<\delta} \frac{1 - \cos ht}{t^2} \frac{(mt - I)t}{(\epsilon + R)^2 + (mt - I)^2} \frac{\sin tx}{t} dt. \tag{11}$$

Since $R(t) \geqq 0$ and $R(t) = o(t)$, $I(t) = o(t)$ as $t \to 0$, we have

$$\frac{|(mt - I)t|}{(\epsilon + R)^2 + (mt - I)^2} \leqq \frac{C_1 t^2}{R^2 + (mt - I)^2} \leqq C_2$$

where C_1, C_2, and so forth denote absolute constants. Hence we can first let $\epsilon \to 0$ under the integral sign in (11); then let $x \to \infty$ and we get, by standard argument, as the value of (11):

$$\lim_{t\to 0} \frac{1 - \cos ht}{t^2} \frac{(mt - I)t}{R^2 + (mt - I)^2} = \frac{h^2}{2m}.$$

Thus in order to prove (10) it remains to show that

$$\lim_{\delta\to 0} \lim_{x\to\infty} \lim_{\epsilon\to 0} \frac{1}{\pi} \int_{|t|<\delta} \frac{1 - \cos ht}{t^2} \frac{(\epsilon + R) \cos tx}{(\epsilon + R)^2 + (mt - I)^2} dt = \frac{h^2}{2|m|}. \tag{12}$$

Let

$$H = H(t) = (1 - \cos ht)/t^2, \quad A = A(t) = mt - I(t).$$

The integral in (12) can be written as

$$H\frac{(\epsilon + R)(\cos tx - 1)}{(\epsilon + R)^2 + A^2} + H\frac{\epsilon}{(\epsilon + R)^2 + A^2} + H\frac{R}{(\epsilon + R)^2 + A^2}.$$

Call these three parts J_1, J_2, and J_3 respectively.

We need the following lemma.

LEMMA. *$R/(R^2+A^2)$ is integrable in $|t| < \delta$.*

PROOF. Since $A^2 \sim m^2t^2$ as $t \to 0$, we have

$$\frac{R}{R^2 + A^2} \leqq C_3 \frac{R}{t^2}.$$

By definition we have

$$\int_{|t|<\delta} \frac{R}{t^2}\, dt = \int_{-\delta}^{\delta} \frac{dt}{t^2} \int_{-\infty}^{\infty} (1 - \cos tx) dF(x).$$

Since the integrand is positive, we can invert the double integral and get

$$\begin{aligned} \int_{-\infty}^{\infty} dF(x) \int_{-\delta}^{\delta} \frac{1 - \cos tx}{t^2}\, dt &= \int_{-\infty}^{\infty} |x|\, dF(x) \int_{-\delta|x|}^{\delta|x|} \frac{1 - \cos t}{t^2}\, dt \\ &\leqq \int_{-\infty}^{\infty} |x|\, dF(x) \int_{-\infty}^{\infty} \frac{1 - \cos t}{t^2}\, dt < \infty. \end{aligned}$$

5°. We have, by the lemma,

$$\begin{aligned} \lim_{\delta\to 0} \lim_{x\to\infty} \lim_{\epsilon\to 0} \int_{|t|<\delta} J_3 dt &= \lim_{\delta\to 0} \lim_{\epsilon\to 0} \int_{|t|<\delta} H \frac{R}{(\epsilon + R)^2 + A^2}\, dt \\ &= \lim_{\delta\to 0} \int_{|t|<\delta} H \frac{R}{R^2 + A^2}\, dt = 0. \end{aligned} \tag{13}$$

Next, we notice that

$$\frac{|\cos tx - 1|}{(\epsilon + R)^2 + A^2} \leqq \frac{C_4 t^2}{R^2 + A^2} \leqq C_5$$

for fixed x and $|t| < \delta$. Hence

$$\lim_{\epsilon\to 0} \int_{|t|<\delta} J_1 dt = \int_{|t|<\delta} H \frac{(\cos tx - 1)R}{R^2 + A^2}\, dt.$$

Now let $x \to \infty$, by the Riemann-Lebesgue lemma and the lemma above we get as limit

$$\int_{|t|<\delta} H \frac{-R}{R^2 + A^2} dt.$$

Finally let $\delta \to 0$ and we get 0. Hence

$$\lim_{\delta \to 0} \lim_{x \to \infty} \lim_{\epsilon \to 0} \int_{|t|<\delta} J_1 dt = 0. \tag{14}$$

It remains to consider J_2 and to show that

$$\lim_{\delta \to 0} \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|t|<\delta} H \frac{\epsilon}{(\epsilon + R)^2 + A^2} dt = \frac{h^2}{2|m|}.$$

By standard argument we have

$$\lim_{\delta \to 0} \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|t|<\delta} H \frac{\epsilon}{\epsilon^2 + m^2 t^2} dt = \frac{h^2}{2|m|}. \tag{15}$$

Therefore we must show that

$$\lim_{\delta \to 0} \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|t|<\delta} H \left\{ \frac{\epsilon}{\epsilon^2 + m^2 t^2} - \frac{\epsilon}{(\epsilon + R)^2 + A^2} \right\} dt = 0. \tag{16}$$

6°. The expression in brackets in (16) is

$$\frac{\epsilon(R^2 + A^2 - m^2 t^2) + 2\epsilon^2 R}{(\epsilon^2 + m^2 t^2)[(\epsilon + R)^2 + A^2]}.$$

Consider the first term and the corresponding integral:

$$\int_{|t|<\delta} H \frac{\epsilon(R^2 + A^2 - m^2 t^2)}{(\epsilon^2 + m^2 t^2)[(\epsilon + R)^2 + A^2]} dt$$

$$\leqq \int_{|t|<\delta} H \frac{\epsilon}{\epsilon^2 + m^2 t^2} \frac{|R^2 + A^2 - m^2 t^2|}{R^2 + A^2} dt.$$

By standard argument the limit of the last integral $\delta \to 0$ is

$$\lim_{t \to 0} H \frac{|R^2 + A^2 - m^2 t^2|}{R^2 + A^2} = 0.$$

Next, consider the second term and the corresponding integral:

$$\int_{|t|<\delta} H \frac{2\epsilon^2 R}{(\epsilon^2 + m^2 t^2)[(\epsilon + R)^2 + A^2]} dt.$$

The integrand is dominated by $2RH/(R^2+A^2)$ which is integrable in $|t|<\delta$ by the lemma, hence we may let $\epsilon\to 0$ under the integral sign and get 0 as limit.

(13)–(16) establish (12), whence the desired result.

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Columbia University and
Cornell University