of our map is a maximal ideal and we see that Statements S2 and S3 are violated. This concludes the proof of our theorem.

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ON ORDERED SKEW FIELDS

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In this paper we shall give a necessary and sufficient condition that a skew field can be ordered; moreover, that the ordering of an ordered skew field $K$ can be extended to an ordering of $L$, $L$ being a given extension of $K$. The first of these two results generalizes to skew fields a theorem of E. Artin and O. Schreier [1],1 according to which a commutative field can be ordered if and only if it is formally real. The second result generalizes in the same sense a recent theorem of J. P. Serre [2].

Our considerations are based on the following definition.

Definition. A skew field is said to be ordered if in its multiplicative group a subgroup of index 2 is marked out which is also closed under addition.

Hence a skew field can be ordered if and only if its multiplicative group has a subgroup of index 2 which is also closed under addition.

We shall now prove the following theorem.

Theorem 1. A skew field $K$ can be ordered if and only if $-1$ cannot be represented as a sum of elements of the form

\[ a_1^2 + a_2^2 + \cdots + a_k^2 \quad (a_i \in K, i = 1, 2, \ldots, k). \]

Remark. This property can be considered as a generalization of the notion "formally real" to the case of skew fields.

The necessity of the condition in Theorem 1 is obvious. In order to prove its sufficiency we consider a skew field $K$ in which $-1$ cannot be represented as a sum of elements (1). We shall show that the

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1 Numbers in brackets refer to the bibliography at the end of the paper.
multiplicative group $K^*$ of $K$ has a subgroup of index 2 which is also closed under addition.

Let $S$ be the subset of all (finite) sums of elements (1) in $K$ with every $a_i \neq 0$. Clearly $0 \notin S$, for in the contrary case we should have a relation

$$-a_1 a_2 \cdots a_k = b_1 b_2 \cdots b_1 + \cdots$$

from which would follow, by multiplication on the right by $a_k^{-2}, \cdots, a_1^{-2}$, that $-1 \in S$ in contradiction to our hypothesis. On the other hand, one can see immediately that

- $s \in S, s' \in S$ imply $ss' \in S$ and $s + s' \in S$,
- $s \in S$ implies $s^{-1} = s \cdot s^{-2} \in S$,
- $s \in S, s \in K^*$ imply $s^{-1} s \in S$.

Hence $S$ is a proper invariant subgroup of $K^*$ which is closed under addition. The order of each element ($\neq 1$) in $K^*/S$ being 2, $K^*/S$ is abelian. Consequently any subgroup $P$ of $K^*$ which contains $S$ is invariant in $K^*$.

Now we define $P$ as a maximal subgroup of $K^*$ for which

(2) $S \subseteq P$, $-1 \notin P$, and $P$ is closed under addition.

The existence of such a group $P$ follows immediately from Zorn’s lemma. We have only to show that the decomposition

(3) $K^* = P \cup (-1)P$

holds. Suppose (3) is not true. Then there exists an element $d$ such that

(4) $d \in K^*$, $d \in P$, $-d \notin P$.

Consider the set $P'$ of all elements

$$u + vd \quad (u, v \in \{P, 0\} \text{ but not } u = v = 0).$$

Then, by (4), $P'$ contains $P$ as a proper subset. On the other hand we shall show that $P'$ is a subgroup of $K^*$ having the properties (2) (with $P'$ instead of $P$). This is a contradiction to the maximal property of $P$, which will complete the proof.

First we show that $0 \in P'$. Indeed, by the exclusion of $u = v = 0$, $u + vd = 0$ would imply that $v \neq 0$ and hence that $-d = v^{-1}u \in P$, in contradiction to (4). Moreover, if $u_1 + v_1 d$ and $u_2 + v_2 d$ are arbitrary elements of $P'$, we have
(5) \((u_1 + v_1)(u_2 + v_2) = (u_1u_2 + v_1v_2) + (u_1v_2 + v_1u_2)\).

But since \(P\) is an invariant subgroup of \(K^*\), \(dv_2 = v_1\cdot d\), \(du_2 = u_1\cdot d\) hold with suitable elements \(u_1\), \(v_1 \in P\), so that (5) is an element of \(P'\).

If \(u + vd \in P'\), we obtain
\[(u + vd)^{-1} = (u + vd)(u + vd)^{-1} \in P'.\]

Hence \(P'\) is a group which is obviously closed under addition. Finally, \(-1 \in P'\) for \(u + vd = -1\) would imply (on account of \(v \neq 0\)) that \(-d = v^{-1}(u + 1) \in P\). This completes the proof.

In an analogous manner we prove the following theorem.

**Theorem 2.** Let \(L\) be an extension of the ordered skew field \(K\). The ordering of \(K\) can be extended to an ordering of \(L\) if and only if \(-1\) cannot be represented as a sum of elements

\[(6) \sum_{i=1}^{k} p_i u_i = 0 \quad (p_i \in K, p_i > 0, u_i \in L, i = 1, 2, \ldots, k).\]

**Remark.** Theorem 1 is the special case of Theorem 2 in which \(K\) is the prime field of characteristic zero. However, this special case seemed of sufficient interest to warrant an independent proof. Only a few remarks are now necessary to prove Theorem 2 since the proof follows the same general pattern as that of Theorem 1.

The necessity of the condition in Theorem 2 is obvious. In order to prove its sufficiency we define the subset \(U\) of \(L\) as the set of all (finite) sums of elements (6) with every \(u_i \neq 0\). One can show as above that \(U\) is a subgroup of the multiplicative group \(L^*\) of \(L\). That, e.g., \(0 \in U\) follows from the fact that a relation
\[\sum_{i=1}^{k} p_i u_i = 0 \quad (p_i \in K, p_i > 0, u_i \in L, i = 1, 2, \ldots, k)\]
would imply that
\[\sum_{i=1}^{k} p'_i v_i = \sum_{i=1}^{k} p'_i v_i (1 \cdot u_i) (p^{-1}_k u_{k-1}) \cdots (p^{-1}_1 u_1) + \cdots,\]
which is impossible.

Moreover \(U\) is an invariant subgroup of \(L^*\). This follows from the fact that
\[p \in K, \quad p > 0, \quad z \in L^*\]

imply that
\[z^{-1} p z = z^{-2} z p p z^{-1} = p'_1 v_1, p'_2 v_2, p'_3\]
with \(p'_1 = 1, v_1 = z^{-1}, p'_2 = 1, v_2 = z p, p'_3 = p^{-1}\).
From the fact that each element \( \neq 1 \) of \( L^*/U \) is of order 2, we infer as above that any subgroup \( Q \) of \( L^* \) containing \( U \) is invariant in \( L^* \).

Now we define \( Q \) as a maximal subgroup of \( L^* \) for which \( U \subseteq Q \), \( -1 \in Q \), and \( Q \) is closed under addition. Then one can show as above that \( Q \) is a subgroup of index 2 of \( L^* \). Since all positive elements of \( K \) are contained in \( U \) and consequently in \( Q \), the theorem is proved.


**Bibliography**