HILBERT'S NULLSTELLENSATZ IN INFINITE-DIMENSIONAL SPACE

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1. The theorem. Let k be an algebraically closed field. Let $A = \{\alpha\}$ be an indexing set. Let $\mathfrak{o} = k[x_\alpha]$ be the polynomial ring generated over k by an indexed set of variables x_α, $\alpha \in A$. Let $\mathfrak{a}$ be an ideal of $\mathfrak{o}$. A zero of $\mathfrak{a}$ is a set (ξ_α) of elements ξ_α in some extension field of k such that $f(\xi_\alpha) = 0$ for all $f \in \mathfrak{a}$. (Of course, a polynomial f involves only a finite number of the variables x_α.) A zero of $\mathfrak{a}$ will be called algebraic if all ξ_α lie in k. The set of all algebraic zeros of an ideal $\mathfrak{a}$ will be called the variety defined by $\mathfrak{a}$.

The Hilbert Nullstellensatz is in general not valid if A is an infinite set. We shall prove however the following theorem:

THEOREM. *The following three statements are equivalent*:

S1. *If $\mathfrak{a}$ is an ideal of $\mathfrak{o}$ and $f \in \mathfrak{o}$ vanishes on the variety defined by $\mathfrak{a}$, then $f^\rho \in \mathfrak{a}$ for some integer ρ.*

S2. *If $\mathfrak{a}$ is an ideal of $\mathfrak{o}$ and $\mathfrak{a} \neq \mathfrak{o}$, then $\mathfrak{a}$ has an algebraic zero.*

S3. *A ring extension $k[\xi_\alpha]$ by elements ξ_α in some extension field of k is a field if and only if all ξ_α lie in k.*

Furthermore the three statements hold if and only if one of the following two conditions is satisfied:

(i) *A is a finite set.*

(ii) *Let A have cardinality a. Let the transcendence degree of k over the prime field have cardinality b. Then $a < b$.*

The above two conditions can be replaced by the single condition that the cardinality of the field k itself should be greater than the cardinality of A.[1] However, our proof will apply to the case in which a is finite only under condition (ii) if we interpret $a < b$ to mean that b is infinite.

It is possible to interpret our theorem geometrically. Let k and A satisfy one of our two conditions. Let V_ν be a collection of varieties in the space of $k[x_\alpha]$. If the V_ν have the finite intersection property (that is, finite intersections $V_{\nu_1} \cap \cdots \cap V_{\nu_r}$ are never empty), then $\bigcap_\nu V_\nu$ is not empty. This follows immediately from the fact that the union of all ideals $\mathfrak{a}_\nu$ defining V_ν is not the ring $\mathfrak{o}$ and has therefore an algebraic zero which lies in all varieties V_ν. If neither (i) nor (ii) is satisfied, then the intersection of all V_ν may be empty.

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[1] Professor I. Kaplansky has informed me that he has obtained the theorem independently. His proof is different from the one presented here, however.

When A is a finite set, the above is the usual Hilbert theorem and is well known. A proof was given recently by Zariski [1] to whom Statement 3 is due.

2. **The proof.** We first prove the equivalence of the three statements.

S1→S2. Let $\mathfrak{a}$ be an ideal without algebraic zero. Then the polynomial 1 vanishes (vacuously) on the variety of $\mathfrak{a}$. Hence

$$1 \in \mathfrak{a} \text{ and } \mathfrak{a} = \mathfrak{o}.$$

S2→S1. We reproduce here a well known argument due to Rabinowitsch. The statement is well known if A is a finite set, so we shall assume that A is an infinite set. Let $\mathfrak{a}$ be an ideal of $\mathfrak{o}$ and let f vanish on the variety of $\mathfrak{a}$. Let t be a new variable. In the ring $\mathfrak{o}[t]$ we consider the ideal $(\mathfrak{a}, 1-tf)$. The ring $\mathfrak{o}[t]$ is isomorphic to $\mathfrak{o}$, and $(\mathfrak{a}, 1-tf)$ has no algebraic zero. Hence by S2 there is an expression of the type

$$A_1f_1 + \cdots + A_rf_r + A(1 - tf) = 1$$

where A_ν, $A \in \mathfrak{o}[t]$ and $f_1, \cdots, f_\nu \in \mathfrak{a}$. We may now let $t = 1/f$ and clear denominators to give $f^\rho \in \mathfrak{a}$ for some integer ρ.

S2→S3. Let $k[\xi_\alpha]$ be a field. Let $\mathfrak{p}$ be the kernel of the map $k[x_\alpha] \to k[\xi_\alpha]$ so $\mathfrak{p}$ is a maximal ideal. $\mathfrak{p}$ has an algebraic zero (η_α) and is the kernel of the map $k[x_\alpha] \to k[\eta_\alpha]$. Hence

$$(x_\alpha - \eta_\alpha) \in \mathfrak{p} \text{ (all } \alpha),$$

and therefore $\xi_\alpha = \eta_\alpha$ lies in k.

S3→S2. Let $\mathfrak{a}$ be an ideal $\neq \mathfrak{o}$. Let $\mathfrak{p}$ be a maximal ideal containing $\mathfrak{a}$. Then $\mathfrak{o}/\mathfrak{p}$ is a field $k[\xi_\alpha]$ and all ξ_α lie in k. It follows that (ξ_α) is an algebraic zero of $\mathfrak{a}$.

We shall now prove that the three statements hold precisely under the above-mentioned conditions. We assume familiarity with cardinal arithmetic, and the following fact: If a field F has transcendence degree b over the prime field, and b is not finite, then the set consisting of all elements in the algebraic closure of F has cardinality b also.

If A is a finite set, the theorem is well known. We suppose that A is not finite and that $a < b$. We shall prove that S2 holds, that is: A proper ideal $\mathfrak{a}$ has an algebraic zero.

LEMMA. *If $\mathfrak{a}$ is an ideal of $\mathfrak{o}$, then $\mathfrak{a}$ has a basis consisting of at most a elements.*

PROOF. We consider the set of all subrings $k[x_{\alpha_1}, \cdots, x_{\alpha_r}]$ generated by a finite number of indeterminates. This set has cardinality a, and we may index it by A again. Let these rings be denoted by $\mathfrak{o}_\alpha$ and let $\mathfrak{a}_\alpha = \mathfrak{a} \cap \mathfrak{o}_\alpha$. Then $\mathfrak{a} = \bigcup_\alpha \mathfrak{a}_\alpha$. Each $\mathfrak{a}_\alpha$ has a finite basis and therefore the union of these basis elements is a basis for $\mathfrak{a}$ having cardinality at most a, as was to be shown.

Let $\mathfrak{a}$ be a proper ideal. Let $\mathfrak{p}$ be a prime ideal containing $\mathfrak{a}$. (A maximal such ideal will always exist by Zorn's Lemma.) It has a basis of at most a elements. Let k_0 be the algebraic closure of the field obtained by adjoining to the prime field all coefficients of elements in such a basis of $\mathfrak{p}$. Then k_0 has cardinality at most a, and by hypothesis k/k_0 has transcendence degree of cardinality at least a.

Let $\mathfrak{p}_0 = k_0[x_\alpha] \cap \mathfrak{p}$. It has the same basis as $\mathfrak{p}$. Any algebraic zero of $\mathfrak{p}_0$ will therefore be an algebraic zero of $\mathfrak{p}$. We shall now find an algebraic zero of $\mathfrak{p}_0$. The residue class ring $k_0[x_\alpha]/\mathfrak{p}_0$ is isomorphic to a ring extension $k_0[\xi_\alpha]$ under the natural map, and (ξ_α) is a zero of $\mathfrak{p}_0$. It is easy to see that there exists an isomorphism of $k_0[\xi_\alpha]$ into k which is the identity on k_0. If (ξ'_α) is the image of (ξ_α) under this isomorphism, then (ξ'_α) is the desired algebraic zero. (The above-mentioned isomorphism may be constructed as follows: Consider the field $k_0(\xi_\alpha)$. Select a transcendence base. By our cardinality assumption we may map the rational field generated over k_0 by this base isomorphically into k. It is now well known that this map may be extended to the algebraic closure, and its restriction to the ring $k_0[\xi_\alpha]$ is the desired isomorphism.)

In order to complete the proof we need only show that our conditions (i) and (ii) are necessary. In other words, if neither condition is satisfied, then there exists a proper ideal without algebraic zero. We show how to construct such an ideal.

Suppose that A is an infinite set and that $b \leqq a$. Then the set of elements of k has cardinality $c \leqq a$. Select an element α_0 in A. Let $C = \{\gamma\}$ be a subset of $A - \{\alpha_0\}$ having cardinality c, and index the elements of k by C, so $k = \{\xi_\gamma\}$. Let y be transcendental over k. Map $k[x_\alpha]$ as follows:

$$\begin{aligned} x_\alpha &\to 0, && \alpha \notin C,\ \alpha \neq \alpha_0, \\ x_{\alpha_0} &\to y, && \\ x_\gamma &\to \frac{1}{y - \xi_\gamma}, && \gamma \in C, \end{aligned}$$

and let the map be identity on k. Then $k[x_\alpha]$ is mapped onto $k[y, 1/(y-\xi_\gamma), \cdots]$ which is easily seen to be a field. The kernel of our map is a maximal ideal and we see that Statements S2 and S3 are violated. This concludes the proof of our theorem.

Reference

1. O. Zariski, *A new proof of Hilbert's Nullstellensatz*, Bull. Amer. Math. Soc. vol. 53 (1947) p. 362.

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ON ORDERED SKEW FIELDS

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In this paper we shall give a necessary and sufficient condition that a skew field can be ordered; moreover, that the ordering of an ordered skew field K can be extended to an ordering of L, L being a given extension of K. The first of these two results generalizes to skew fields a theorem of E. Artin and O. Schreier [1],[1] according to which a commutative field can be ordered if and only if it is formally real. The second result generalizes in the same sense a recent theorem of J. P. Serre [2].

Our considerations are based on the following definition.

Definition. A skew field is said to be ordered if in its multiplicative group a subgroup of index 2 is marked out which is also closed under addition.

Hence a skew field can be ordered if and only if its multiplicative group has a subgroup of index 2 which is also closed under addition.

We shall now prove the following theorem.

Theorem 1. *A skew field K can be ordered if and only if -1 cannot be represented as a sum of elements of the form*

$$a_1^2 a_2^2 \cdots a_k^2 \quad (a_i \in K,\ i = 1, 2, \cdots, k). \tag{1}$$

Remark. This property can be considered as a generalization of the notion "formally real" to the case of skew fields.

The necessity of the condition in Theorem 1 is obvious. In order to prove its sufficiency we consider a skew field K in which -1 cannot be represented as a sum of elements (1). We shall show that the

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[1] Numbers in brackets refer to the bibliography at the end of the paper.