A PROBABILISTIC APPROACH TO A SYSTEM OF INTEGRAL EQUATIONS

LEO A. GOODMAN

The following system of integral equations is of some statistical interest:

\[ G_n(y) = \int_0^\infty \frac{e^{-u\omega^{\alpha-1}}}{\Gamma(n\alpha)} \, du = \int_0^{g(y)} H_{n-1}(g(y) - z; g) \, dH_1(z; g) \]

for \( n = 1, 2, \ldots \),

where \( g(x) \geq 0 \) is an increasing continuous function of \( x \geq 0 \), \( H_0(x; g) = 1 \), and

\[ H_n(x; g) = \int_0^x H_{n-1}(x - z; g) \, dH_1(z; g). \]

We shall show by a probabilistic approach (distribution functions, the moment problem, etc.) that the only functions satisfying this system of equations for fixed \( \alpha, 0 < \alpha \leq 2 \), are \( g(x) = cx \), where \( c \) is a constant. It will then be seen that this result is a probabilistic analogue of the well known Cauchy functional equation. Also an application of this result to statistics is presented.

**Theorem 1.** If \( g(x) \) satisfies the Cauchy functional equation

\[ g(x + y) = g(x) + g(y), \]

then \( g(x) \) satisfies (1).

**Proof.** For fixed \( n \), we may define a random variable \( Y \) by the relation \( \Pr \{ Y \leq y \} = G_n(y) \). Since \( H_1(v; g) = \int_0^\infty (e^{-u\omega^{\alpha-1}}/\Gamma(\alpha)) \, du \), where \( f(v) \) is the inverse function of \( g(x) \), we may define a random variable \( V \) by the relation

\[ \Pr \{ V \leq v \} = H_1(v; g). \]

Hence \( H_n(v; g) \) may be regarded as the distribution of the sum of \( n \) independent random variables \( V_i \), each distributed as \( V \). Equation (1) then states

\[ \Pr \{ Y \leq y \} = \Pr \{ V_1 + V_2 + \cdots + V_n \leq g(y) \} = \Pr \{ f(V_1 + V_2 + \cdots + V_n) \leq y \}. \]

Received by the editors April 26, 1951.

1 This research was prepared under an Office of Naval Research contract.

505
When \( n = 1 \), we may write \( Y = f(V_1) \). By the addition theorem for gamma distributions, we have

\[
Pr \{ f(V_1) + f(V_2) + \cdots + f(V_n) \leq y \} = G_n(y)
\]

\[
\Pr \{ f(V_1 + V_2 + \cdots + V_n) \leq y \}.
\]

Since when \( g \) satisfies (2), so does \( f \), a fortiori (4) and (1) are satisfied by \( f \) and \( g \). Q.E.D.

**Theorem 2.** The only functions satisfying (1) for fixed \( \alpha, 0 < \alpha \leq 2 \), are \( g(x) = cx \), where \( c \) is a constant.

**Proof.** We use the notation developed in the proof of Theorem 1. Then \( g(Y) \) is a random variable and by (3) has the same distribution function as the sum of \( n \) independent random variables \( V_i \). Let \( \phi(t) = \int_0^\infty e^{-tx}dG(t(x)) \), for fixed \( t > 0 \). Then by the fundamental property of moment generating functions, \( \int_0^\infty e^{-tx}dG_n(f(x)) = \phi^n(t) \). Hence, we have the system of equations

\[
\int_0^\infty e^{-tx}e^{-u} \frac{\omega^{\alpha-1}}{\Gamma(\alpha)} d\omega = \phi^n(t).
\]

Let us define a new random variable \( Z \) which has the distribution function

\[
Pr \{ Z \leq z \} = \int_0^z \frac{e^{-tx(\omega)-\omega}}{\phi(\alpha)} \omega^{\alpha-1} d\omega, \quad \text{where } \phi = \phi(t).
\]

Hence by (5) we have that the \( n-1 \) moment of \( Z^a/\phi \) is equal to \( \Gamma(n\alpha)/\Gamma(\alpha) \). It can easily be seen that 1, \( \Gamma(2\alpha)/\Gamma(\alpha) \), \( \Gamma(3\alpha)/\Gamma(\alpha) \), \( \cdots \) is the sequence of moments of the density function

\[
e^{-\frac{a}{\Gamma(\alpha+1)}} dh.
\]

By (6), the density function of \( Z^a/\phi = k \) is

\[
e^{-\frac{\omega(\phi k)^{1/\alpha} - (\phi k)^{1/\alpha}}{\Gamma(\alpha+1)}} dk.
\]

Since the moment problem is determined for \( \alpha \leq 2 \) (cf. [1]), we have that

\[
k^{1/\alpha} \left[ \frac{1 - \phi^{1/\alpha}}{t} \right] = g((\phi k)^{1/\alpha})
\]

or
We now see, combining the preceding theorems, that in the case under consideration, if the Cauchy functional equation (2) is satisfied, then the function must be \( g(x) = cx \), where \( c \) is a constant.

The following statistical result may be proved using the preceding theorem.

**Theorem 3.** Let \( X_1, X_2, \ldots \) be a sequence of non-negative independent random variables with the same continuous distribution function \( F(x) \), and let \( N_x \) be defined as follows:

\[
N_x = \begin{cases} 
0 & \text{if } X_1 > x, \\
 n & \text{if } X_1 + X_2 + \cdots + X_n \leq x \\
 & \text{and } X_1 + X_2 + \cdots + X_{n+1} > x.
\end{cases}
\]

If \( N_x \) has a distribution function of the form

\[
\Pr \{ N \leq n \} = e^{-f(x)} \left[ 1 + f(x) + \frac{1}{2!} f^2(x) + \cdots \right. \\
\left. \quad + \frac{1}{(n + 1)!} f^{(n+1)}(x) \right],
\]

with fixed \( \alpha = 1 \) or 2, for every positive \( x \), then \( F(x) \) is of the gamma type

\[
F(x) = \begin{cases} 
0 & \text{for } x < 0, \\
\int_0^x \frac{\omega^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} e^{-\omega/\beta} d\omega & \text{for } x \geq 0.
\end{cases}
\]

Mr. Seiji Nabeya [2] has obtained a proof for the case \( \alpha = 1 \) when \( F(x) \) is not assumed to be continuous.

**References**


**The University of Chicago**