TREES, LATTICES, ORDER, AND BETWEENNESS[1]

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In this paper we consider postulates expressed in terms of "segments," "medians," and "betweenness." Characterizations are obtained for trees, lattices, and partially ordered sets. In general a characterization is given by a system of three postulates. These systems fall in pairs; systems of a pair have two postulates in common. An algebra which has both lattices and trees as special cases is given in the final section.

1. **Segments.** Consider a set S of elements $a, b, c, \cdots$ such that to each pair a, b, of elements in S there corresponds a unique subset of S denoted by (a, b) and called the segment from a to b. By assumption, these segments have as properties:

(S) To each set of three elements a, b, and c, there corresponds an element d such that $(a, b)\cap(b, c)=(b, d)$.

(T) $(a, b)\subset(a, c)$ implies $(a, b)\cap(b, c)=\{b\}$.

The segments of Duthie [1][2] are segments in this sense (see §3). Paths in a tree are also segments in this sense (see §2).

Setting $a=b=c$ in (T) we have

$$\{a\} = (a, a). \tag{1.1}$$

From this, and from (T) with $b=c$, we have

$$b \in (a, b). \tag{1.2}$$

$$a \in (a, b). \tag{1.3}$$

Proof. From (S), we may choose d so that $(a, a)\cap(a, b)=(a, d)$. By (1.1) and (1.2), $d\in(a, d)\subset(a, a)=\{a\}$. Hence $d=a$ and $a\in(a, d)\subset(a, b)$.

$$(a, b) = (b, a). \tag{1.4}$$

Proof. From (S), we may choose d so that $(a, b)\cap(b, a)=(b, d)$. By (1.2) and (1.3), $a\in(b, d)$. By (T), $(b, d)\cap(d, a)=\{d\}$. Hence $a\in\{d\}$, $a=d$, and $(b, a)=(b, d)\subset(a, b)$. By symmetry, $(a, b)\subset(b, a)$.

$$b \in (a, c) \text{ if and only if } (a, b) \subset (a, c). \tag{1.5}$$

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

PROOF. The sufficiency proof is trivial. To prove necessity choose d so that $(b, a)\cap(a, c)=(a, d)$. Clearly $b\in(a, d)$. From (1.4) and (T), $(a, d)\cap(d, b)=\{d\}$. Hence $b\in\{d\}$, $b=d$, and $(a, b)=(a, d)\subset(a, c)$.

From (1.5) and (T) we have

(1.6) $b \in (a, c)$ implies $(a, b) \cap (b, c) = \{b\}$.

(1.7) $b \in (a, c)$ and $c \in (a, b)$ imply $b = c$.

Thus if $(a, b)=(a, c)$ we have $b=c$.

PROOF. By (1.5), $(a, b)\subset(a, c)\subset(a, b)$. Hence these segments are equal. From (1.4) and (1.6), $\{b\}=(a, b)\cap(b, c) =(a, c)\cap(c, b)=\{c\}$.

As a corollary of (1.7) we have

(1.8) The element d of (S) is unique.

(1.9) $b\in(a, c)$ if and only if $(a, b)\cup(b, c)\subset(a, c)$.

PROOF. The condition is necessary by (1.4) and (1.5). It is sufficient by (1.2) and (1.3).

(1.10) $(a, b) \cap (b, c)=(b, d)$ implies $(a, d) \cap (d, c) = \{d\}$.

PROOF. We note $d\in(a, b)$ and $d\in(b, c)$. From (1.9), $(a, b)\supset(a, d)\cup(d, b)$ and $(b, c)\supset(b, d)\cup(d, c)$. Taking set intersections we obtain $(b, d)\supset(b, d)\cup[(a, d)\cap(d, c)]$. From this and from (1.6), $(a, d)\cap(d, c)=(a, d)\cap(d, c)\cap(b, d)=\{d\}\cap(d, c)=\{d\}$.

2. **Tree segments.** The word tree is probably most often used in mathematics to denote a finite connected linear graph which contains no cycles [**2**, p. 47]. However a connected acyclic union of closed Jordan arcs is sometimes called a tree and in *Lattice theory* [**3**, p. 47] we find the word used in still a third sense. Seeking characteristics common to the several types of objects known as trees we arrive at the definition below. (Birkhoff's trees can be imbedded in our trees. Trees in our sense which are finite are trees in König's sense.)

A tree is defined as a set of elements which satisfies (S), (T), and:

(U_1) $(a, b)\cap(b, c)=\{b\}$ implies $(a, b)\cup(b, c)=(a, c)$.

From this definition, (1.9), and (1.6) we have

(2.1) $b\in(a, c)$, $(a, b)\cap(b, c)=\{b\}$, and $(a, b)\cup(b, c)=(a, c)$ are equivalent conditions.

(2.2) $(a, b)\cap(b, c)=(b, d)$ if and only if $\{d\}=(a, b)\cap(b, c)\cap(c, a)$. Hence the last term always represents a set containing a single point.

PROOF. If $(a, b)\cap(b, c)=(b, d)$, we have by (1.10) that $(a, d)\cap(d, c)=\{d\}$. Using (2.1) and the distributive law, we have (a, b)

$\cap(b, c)\cap(c, a)=(b, d)\cap(c, a)=(b, d)\cap[(a, d)\cup(d, c)]=\{d\}\cup\{d\}$ $=\{d\}$. The sufficiency of the condition follows from (S) and the necessity of the condition.

DEFINITION. The unique element given by $(a, b)\cap(b, c)\cap(c, a)$ is called the median of a, b, and c and is denoted by (a, b, c). Properties of this ternary operation are derived in §5.

(2.3) For all a, b, and c, $(a, b)\subset(a, c)\cup(c, b)$.

PROOF. Let $d=(a, b, c)$. From (2.1), $(a, b)=(a, d)\cup(d, b)$. From (1.5), $(a, d)\subset(a, c)$ and $(d, b)\subset(c, b)$.

DEFINITION. We say b is between a and c and write abc if and only if $b\in(a, c)$.

As a consequence of (1.1) we have

(2.4) aba if and only if $a=b$.

As a consequence of (2.2) we have

(2.5) To elements a, b, and c there corresponds a unique element d such that adb, bdc, and cda.

(2.6) If we have both abc and bde, then we have either cbd or eba, perhaps both.

PROOF. Let $b\in(a, c)$ and $d\in(b, e)$. By (2.3), if $b\notin(a, e)$ then $b\in(e, c)$. Again, if $b\notin(c, d)$ then $b\in(d, e)$. But from (1.7) we have in this case the contradiction $b=d\in(c, d)$.

3. **Lattice segments.** Consider a set S satisfying (S), (T), and:

(U_2) There are elements 0 and I in S such that if $(0, r)\cap(0, s)$ $\subset(0, a)\cap(0, b)$ and $(I, r)\cap(I, s)\subset(I, a)\cap(I, b)$, then $(a, b)\subset(r, s)$.

(3.1) $r\in(0, a)$ if and only if $a\in(r, I)$.

PROOF. Let $b=s=I$ in (U_2) to prove necessity. The proof of sufficiency is dual.

Letting $r=0$ in (3.1), we have

(3.2) For all a, $a\in(0, I)$.

(3.3) If $(0, a)\cap(0, b)=(0, r)$ and $(I, a)\cap(I, b)=(I, s)$ then $(a, b)=(r, s)$.

PROOF. From $s\in(I, a)\cap(I, b)$ and from (3.1) we have that a and $b\in(0, s)$. Hence $(0, r)=(0, a)\cap(0, b)\subset(0, s)$. By (3.1), $(I, s)\subset(I, r)$. A double application of (U_2) gives $(a, b)\subset(r, s)$ and $(r, s)\subset(a, b)$.

DEFINITION. We write $r=ab$ if and only if $(0, r)=(0, a)\cap(0, b)$.

We write $s=a+b$ if and only if $(I, s)=(I, a)\cap(I, b)$.

(3.4) S is a lattice with bounds 0 and I.

PROOF. The commutative, associative, and idempotent laws are easily derived. Properties of 0 and I follow easily from (3.2). In proving an absorption law, we let $a+b=c$ and $ac=d$. From $(I, c)=(I,a)\cap(I, b)$ and (3.1), we have a in $(0, c)$. Hence $(0, a)=(0, a)\cap(0, c)=(0, d)$. By (1.7), $a=d$. The other absorption law follows dually.

(3.5) $x \in (a, b)$ if and only if $ab \leqq x \leqq a + b$.

PROOF. By (3.3), $x\in(r, s)$ where $r=ab$ and $s=a+b$. As in the proof of (3.3), $(0, r)\subset(0, s)$. By (1.5), $(r, s)\subset(0, s)$. By (3.1), $s\in(I, x)$. Hence $(I, s)\subset(I, x)$ and $x+s=s$. Dually, $xr=r$. To prove the inequality sufficient, we note $xr=r$ implies $(0, r)\subset(0, x)$. Similarly, $(I, s)\subset(I, x)$. Hence $(0, a)\cap(0, b)\subset(0, a)\cap(0, x)$ and $(I, a)\cap(I, b)\subset(I, a)\cap(I, x)$. By (U_2), $(a, x)\subset(a, b)$ and $x\in(a, b)$.

(3.6) Postulates S, T, and U_2 characterize distributive lattices with 0 and I.

PROOF. That these postulates give a distributive lattice with 0 and I follows from (3.4), (3.5), (1.6), and a theorem of Duthie [1]. Conversely, if in such a distributive lattice we define (a, b) as the set of all x such that $ab\leqq x\leqq a+b$, the three postulates are easily derived.

4. **Betweenness.** We now consider a set S of elements $a, b, c, \cdots$ in which, for each ordered set of three elements a, b, and c, there holds or fails to hold a relation denoted by abc and read "b is between a and c." This relation satisfies the following postulates:

(B) $aba\leftrightarrow a=b$.

(C) $abc\cdot bde\rightarrow(cbd$ or $eba)$.

For the interpretation of the notation in (C) see (2.6). We proceed to derive consequences of these two postulates.

(4.1) $aab \rightarrow baa$.

PROOF. By (C), $aab\rightarrow aab\cdot aab\rightarrow(baa$ or $baa)\rightarrow baa$.

(4.2) $abc \rightarrow aab$.

PROOF. If $a=c$, $abc\rightarrow aba\rightarrow a=b$. But by (B) we have aaa, and now $aaa\rightarrow aab$. If $a\neq c$, from $aaa\cdot abc\rightarrow(aab$ or $caa)$, we have either aab or $caa\cdot abc\rightarrow(aab$ or $cac)\rightarrow aab$.

(4.3) $abc \rightarrow cba$.

PROOF. If $a=b$, this follows from (4.1). If $a\neq b$, we note by (4.2) and (4.1) that $abc\rightarrow baa$ and we then have that $abc\cdot baa\rightarrow(cba$ or $aba)$ $\rightarrow cba$.

(4.4) Each of the relations aab, abb, bba, and baa implies the other three.

PROOF. By (4.1), $aab\rightarrow baa$ and $bba\rightarrow abb$. By (4.2), $baa\rightarrow bba$ and $abb\rightarrow aab$.

DEFINITION. We say a is comparable with b and write ab if and only if aab holds. It follows that aa holds for all a in S, that $ab\rightarrow ba$, and that ab is equivalent to each of the betweenness relations of (4.4). It is unlikely that this notation will be confused with the product notation of §3.

(4.5) $$abc \rightarrow ab\cdot bc\cdot ca.$$

PROOF. By (4.2) and (4.3), $abc\rightarrow ab$ and $abc\rightarrow cba\rightarrow cb$. If $a=b$, $cb\rightarrow ca$. If $a\neq b$, by (4.2) and (C), we have $abc\rightarrow aab\cdot abc\rightarrow(bab$ or $caa)$ $\rightarrow caa\rightarrow ca$.

DEFINITION. We say $a_1, a_2, \cdots, a_n$ form a chain and denote this by $a_1a_2\cdots a_n$ if and only if $a_ia_ja_k$ holds for $1\leqq i\leqq j\leqq k\leqq n$. We note the definition is consistent with our previous notation when $n=2, 3$. Clearly, $a_1a_2\cdots a_n$ implies both $a_na_{n-1}\cdots a_1$ and $a_ia_{i+1}\cdots a_j$ for $1\leqq i\leqq j\leqq n$. Moreover when $a_i=a_j$ we have $a_i=a_{i+1}=\cdots=a_{j-1}=a_j$.

(4.6) $$abc\cdot bcd\cdot b\neq c\rightarrow abcd.$$

PROOF. When $b\neq c$ we have $abc\cdot bcd\rightarrow(cbc$ or $dba)\rightarrow abd$. Similarly, $dcb\cdot cba\rightarrow acd$.

(4.7) $$abc\cdot acd\rightarrow abcd.$$

PROOF. If $a=c$, $abc\rightarrow aba\rightarrow a=b$ and the implication holds. If $a\neq c$, $acd\cdot cba\rightarrow(dcb$ or $aca)\rightarrow dcb$. If $b=c$, $acd\rightarrow abd$. Finally we have abd when $b\neq c$ by (4.6).

An easy induction proof establishes the following generalization of (4.6) and (4.7).

(4.8) $$a_1a_2\cdots a_n\cdot a_1a_nb\rightarrow a_1a_2\cdots a_nb,$$
$$a_1a_2\cdots a_n\cdot a_{n-1}a_nb\cdot a_{n-1}\neq a_n\rightarrow a_1a_2\cdots a_nb,$$

and for $1\leqq i\leqq n-1$,

$$a_1a_2\cdots a_n\cdot a_iba_{i+1}\rightarrow a_1a_2\cdots a_iba_{i+1}\cdots a_n.$$

Finally, we may easily prove

(4.9) $abc \cdot acb \rightarrow b = c.$

(4.10) $abc \cdot bd \rightarrow (abd \text{ or } cbd).$

It may be noted that we have made no attempt to use the intensive survey of betweenness made by Pitcher and Smiley [4] because of their initial assumption that every pair of elements is comparable. (This follows from their Postulate β.) It is interesting, however, that (B) and (C) imply, in addition to Postulate α and (1) and (2), 40 of the 43 transitivities given in Part I of their paper.[3] The three transitivities not implied are their T_4, T_7, and T_{10}. Transitivity T_{10} holds, however, in order betweenness (see (6.1)), and T_4 and T_7 hold in tree betweenness (see (8.10) and (8.11)). That T_4 and T_7 do not hold in order betweenness (without the restriction $a \neq b$) exposes a minor error in a comment by Pitcher and Smiley [4, footnote 4].

5. **Tree betweenness.** Consider a set S which satisfies (B), (C), and:

(D_1) Given a, b, and c, there exists an x such that $axb \cdot bxc \cdot cxa$.

We show in (5.8) that S is a tree.

First, from (D_1) and (4.5), it follows that every pair of elements is comparable.

(5.1) For a, b in S, ab holds.

(5.2) $axb \cdot ayb \rightarrow (axyb \text{ or } ayxb).$

Proof. From (5.1), xy holds. From (4.10), $axb \cdot xy \rightarrow (axy \text{ or } bxy)$. Assume, say, axy holds. By (4.7), $axyb$ holds.

(5.3) The element x in (D_1) is unique.

Proof. Assume y has the same property. From $axb \cdot ayb$ and (5.2) we have, say, $axyb$ and hence $axy \cdot xyb$. By (4.7), we have $axyc \cdot cxyb$. From xyc, cxy, and (4.9), $x = y$.

Definitions. The element x of (D_1) is called the median of a, b, and c and is denoted by (a, b, c). The set of all x such that axb holds is called the segment from a to b and is denoted by (a, b).

(5.4) $(a, b) \subset (a, c)$ implies $(a, b) \cap (b, c) = \{b\}$.

Proof. We have given $axb \rightarrow axc$. We are to show $y = b$ if and only if $ayb \cdot byc$. Necessity of the condition follows from (5.1). Conversely, noting ab holds and using (4.8), we have $abb \rightarrow abc \rightarrow aybyc \rightarrow y = b$.

(5.5) $(a, b) \cap (b, c) = \{b\}$ implies $b \in (a, c)$.

[3] The stronger pair of postulates, (C) and their (β), imply 42 of the 43 transitivities, all except T_{10}.

PROOF. We are to show that if $axb \cdot bxc \rightarrow x = b$, then abc holds. This follows by choosing $x = (a, b, c)$.

(5.6) $(a, b) \cap (b, c) = \{b\}$ implies $(a, b) \cup (b, c) = (a, c)$.

PROOF. Assume $axb \cdot bxc \rightarrow x = b$. To show ayc holds if and only if we have either ayb or byc. By (5.5), abc holds. The conclusion follows from (4.7) and (5.2).

(5.7) If $d = (a, b, c)$, then $(a, b) \cap (b, c) = (b, d)$.

PROOF. To show bxd holds if and only if $axb \cdot bxc$. Necessity of the condition is easily seen. Conversely, from (5.2),

$$axb \rightarrow (axdb \text{ or } adxb) \rightarrow (axd \text{ or } dxb),$$

$$bxc \rightarrow (bxdc \text{ or } bdxc) \rightarrow (bxd \text{ or } dxc).$$

If bxd does not hold, we have from (4.8) the contradiction

$$adc \cdot axd \cdot dxc \rightarrow axdxc \rightarrow x = d.$$

(5.8) Trees are characterized as sets S satisfying Postulates B, C, and D_1.

PROOF. This follows from (2.4), (2.5), (2.6), (5.4), (5.6), and (5.7).

For later use we derive properties of the median. As a consequence of its definition, (a, b, c) is invariant under cyclic permutations of a, b, and c. Hence from (4.3), we have the following.

(5.9) (a, b, c) is invariant under all permutations of a, b, and c.

As a consequence of (4.4) and (5.1) we have

(5.10) $(a, x, b) = x$ if and only if axb holds. Thus $(a, a, b) = a$.

(5.11) $((x, a, b), c, x) = (x, a, c)$ or (x, b, c).

PROOF. Let $y = (x, a, b)$ and $z = (y, c, x)$. By (4.7), $xzy \cdot xya \rightarrow xzya \rightarrow xza$. Similarly, we have xzb. Since xzc holds, it remains to show that either azc or bzc holds. If $y = z$, from (5.1) and (4.10) we have $azb \cdot zc \rightarrow (azc \text{ or } bzc)$. If $y \neq z$, we note $xzya \rightarrow ayz$ and from (4.6) obtain $ayz \cdot yzc \rightarrow ayzc \rightarrow azc$.

(5.12) $((a, b, c), (a, b, d), e) = ((c, d, e), a, b)$.

PROOF. Let $x = (a, b, c)$, $y = (a, b, d)$, $r = (x, y, e)$, $z = (c, d, e)$, and $s = (a, b, z)$. We are to prove $r = s$. From axb, ayb, and (5.2), we have $axyb$ or $ayxb$. Since these cases are handled similarly we assume:

(*) $axyb$ holds.

Case I. $x \neq r$ and $r \neq y$.

We have from xry that $x \neq y$. Using (*), we have

$$byx \cdot bxc \to byxc \to cxy,$$

$$axy \cdot ayd \to axyd \to xyd,$$

$$cxy \cdot xyd \cdot xry \to cxryd \to crd,$$

$$cxryd \cdot ery \to eryd \to erd,$$

and

$$cxryd \cdot erx \to cxre \to cre.$$

But $crd \cdot dre \cdot erc$ implies $r = (c, d, e) = z$. Moreover, from (*) and xry, we have arb. Hence from (5.10) we have $r = (a, r, b) = (a, z, b) = s$.

Case II. $x \neq r$, $r = y$.

We omit the proof since it is similar to the proof of Case III.

Case III. $x = r$, $r \neq y$.

We may assume $z \neq x$ for otherwise from axb we have $r = x = (a, b, x) = s$. If czx holds we have $czx \cdot cxb \to zxb$ and, similarly, zxa. Then $axb \cdot bxz \cdot zxa$ imply $r = x = (a, b, z) = s$. It remains to show that to assume czx does not hold leads to a contradiction. Thus

$$czd \cdot zx \to (czx \text{ or } dzx) \to dzx,$$

$$cze \cdot zx \to (czx \text{ or } ezx) \to ezx,$$

$$ezx \cdot exy \to ezxy \to zxy,$$

and

$$dzx \cdot zxy \cdot z \neq x \to dzxy \to dxy.$$

Finally, using (*), $axy \cdot ayd \to xyd$. From $dxy \cdot xyd \cdot x \neq y \to dxyd$ we have the contradiction $x = d = y$.

Case IV. $x = r = y$.

To show $x = (a, b, c) = (a, b, d)$ and $z = (c, d, e)$ imply $x = (a, b, z)$. We note $czd \cdot zx \to (czx$ or $dzx)$. The cases are similarly treated. Assume, say, that czx holds. Then $czx \cdot cxa \to zxa$ and $czx \cdot cxb \to zxb$. Since axb holds, $r = x = (a, b, z) = s$.

6. **Order betweenness.** Consider a set S satisfying Postulates B, C, and:

(D_2) For odd $n \geqq 3$, $a_1a_2 \cdot a_2a_3 \cdots a_{n-1}a_n \cdot a_na_1$ implies either $a_{n-1}a_na_1$, $a_na_1a_2$, or $a_ia_{i+1}a_{i+2}$ for some i, $1 \leqq i \leqq n-2$.

As a typical application of (D_2) we sketch the proof of transitivity T_{10} of Pitcher and Smiley [4].

(6.1) $$abc \cdot abd \cdot xbc \cdot a \neq b \cdot b \neq c \rightarrow xbd.$$

PROOF. The hypotheses imply $cx \cdot xb \cdot bd \cdot da \cdot ac$. By ($D_2$), we have *cxb*, *xbd*, *bda*, *dac*, or *acx*. If *cxb* or *bda* holds, then b equals x or d and *xbd* holds. If *dac* or *acx* holds, we easily derive as a contradiction that b equals a or c.

In view of (B), (4.3), (4.2), (4.6), and (D_2), it is clear that the following theorem is a corollary of Altwegg's results [**5**, Sect. 2].

(6.2) Postulates B, C, and D_2 characterize partially ordered sets to within dual orderings of their connected subsets. Here *abc* is equivalent to either $a \leqq b \leqq c$ or $c \leqq b \leqq a$.

It is possible to characterize a lattice in terms of betweenness by adding a fourth postulate to (B), (C), and (D_2). Then *abc* holds if and only if either $a+b=bc$ or $ab=b+c$. This is, of course, not the betweenness ordinarily called lattice betweenness. The latter, for distributive lattices, has *abc* equivalent to the condition $ac+b = b(a+c)$.

7. **Chain betweenness.** A chain in the sense used here is often called a completely ordered set or a linearly ordered set. An extensive literature is devoted to discussion of chain betweenness.

Consider the postulate:

(D_3) For a, b, and c in S either *abc*, *bca*, or *cab* holds.

(7.1) Chains are characterized by Postulates B, C, and D_3 or by Postulates B, C, D_1, and D_2.

PROOF. The first statement follows from a result of Altwegg [**5**, Sect. 4] and from (D_3), (4.3), (B), (4.6), and (4.7). Since a chain is clearly both a tree and a partially ordered set, it remains to be proved only that (B), (C), (D_1), and (D_2) imply (D_3). This follows from (5.1) and (D_2), since $ab \cdot bc \cdot ca$ implies *abc*, *bca*, or *cab*.

8. **Medians.** Some of the work in this section is similar to previous work (see, for example, [**3**, p. 137]). It has not, however, been previously shown that the results can be made independent of Postulate O_2 below.

Consider a set S closed under a ternary operation (a, b, c), called the median of a, b, and c, satisfying the following postulates:

(M) $(a, a, b) = a$.

(N) $((a, b, c), (a, b, d), e) = ((c, d, e), a, b)$.

The following theorem is proved by setting $c=d$ in (N) and applying (M). We have (8.2) as a corollary.

(8.1) $$(a, b, c) = (c, a, b).$$

(8.2) $$(b, a, a) = (a, b, a) = (a, a, b) = a.$$

The next theorem is used freely in the work to follow.

(8.3) The median (a, b, c) is invariant under permutations of a, b, and c.

PROOF. In view of (8.1), it is sufficient to show $(a, b, c) = (b, a, c)$. This follows from the equalities

$$\begin{aligned}(a, b, c) &= ((b, a, a), (b, a, b), c) = ((a, b, c), b, a)\\ &= ((a, b, c), (a, b, b), a) = ((c, b, a), a, b)\\ &= ((b, a, c), a, b) = ((a, b, b), (a, b, a), c) = (b, a, c).\end{aligned}$$

The next theorem is a direct consequence of (N) and (8.2).

(8.4) $$((a, b, c), a, b) = (a, b, c).$$

(8.5) $$((a, x, b), (b, x, c), (c, x, a)) = ((x, a, c), b, x).$$

PROOF. By (N) and (8.4) both expressions equal $((a, c, (c, x, a)), b, x)$. As a corollary we have

(8.6) $((x, a, c), b, x)$ is invariant under permutations of a, b, and c.

(8.7) $$((a, b, c), (a, b, d), (a, b, e)) = ((c, d, e), a, b).$$

PROOF. Using (N) and (8.6), we have

$$\begin{aligned}&((a, b, c), (a, b, d), (a, b, e))\\ &\quad = (((a, b, d), (a, b, e), a), ((a, b, d), (a, b, e), b), c)\\ &\quad = (((d, e, a), a, b), ((d, e, b), a, b), c)\\ &\quad = (((d, b, a), a, e), ((d, a, b), e, b), c)\\ &\quad = ((a, b, c), (d, b, a), e) = ((c, d, e), a, b).\end{aligned}$$

DEFINITION. We say x is between a and b and write axb if and only if $x = (a, x, b)$. Extensions of this notation such as $abcd$ are made as in §4.

The next two theorems are immediate.

(8.8) $$abc \rightarrow cba$$

(8.9) $$abc \cdot acb \leftrightarrow b = c.$$

Hence aab holds and $aba \leftrightarrow a = b$.

(8.10) $$abc\cdot bcd\cdot ade \rightarrow bce.$$

PROOF. We have $(b, c, e)=((a, b, c), (b, c, d), e)=((a, d, e), b, c) =(d, b, c)=c$.

(8.11) $$abc\cdot abd\cdot ced \rightarrow abe.$$

PROOF. We have $(a, b, e)=(a, b, (c, d, e))=((a, b, c), (a, b, d), e) =(b, b, e)=b$.

The implications just established are found in [4] under the labels α, β, T_4, and T_7. We have from §6 of that paper the following as consequence.

(8.12) $$abc\cdot acd \rightarrow abcd$$

(8.13) Given a, b, and c, there is a unique element x such that $axb\cdot bxc\cdot cxa$.

PROOF. From (8.4), (a, b, c) satisfies these betweenness relations. If the relations hold for both x and y then $x=(x, x, y)=((a, x, b), (a, x, c), y)=((b, c, y), a, x)=(y, a, x)$ and, similarly, $y=(y, a, x)$.

If we define (a, b) as the set of all x such that $x=(a, x, b)$ it is not difficult to prove the following.

(8.14) Postulates M and N imply Postulates S and T.

9. **Tree medians.** Consider Postulates M, N, and:

(O_1) $((x, a, b), c, x)=(x, a, c)$ or (x, b, c).

It is clear from (8.6) and (8.7) that this postulate can be restated as follows. The elements (a, x, b), (b, x, c), and (c, x, a) are not distinct.

We wish to show these postulates characterize trees. By (5.10), (5.11), and (5.12), Postulates B, C, and D_1 imply Postulates M, N, and O_1. Conversely, by (8.9) and (8.13), (M), (N), and (O_1) imply (B) and (D_1). To show (C) also holds let $b=(a, b, c)$, $d=(b, d, e)$, and $b\neq(c, b, d)$. By (8.2), $b\neq d$. By (O_1), $b=(b, d, b)=((b, c, a), d, b) =(b, a, d)$. Hence $b=((b, a, d), e, b)=(b, a, e)$ or (b, d, e). To assume the latter gives the contradiction $b=d$. Hence $b=(b, a, e)$ and eba holds. By (5.8), the following holds.

(9.1) Postulates M, N, and O_1 characterize trees.

10. **Lattice medians.** Consider Postulates M, N, and:

(O_2) There are elements 0 and I in S such that for all a in S, $a=(0, a, I)$.

In view of (8.3), the following is a restatement of a known theorem [3, p. 137].

(10.1) Distributive lattices with bounds 0 and I are characterized by Postulates M, N, and O_2. This is achieved by use of the relations $ab=(a, 0, b)$, $a+b=(a, I, b)$, and $(a, b, c)=ab+bc+ca$.

It is possible to characterize these lattices by means of only two postulates based on medians (see [6]).

11. **Chain medians.** Results in this section are similar to those in §7. We consider the postulate

(O_3) If $(a, b, c)\neq a$ or c, then for all x either (a, x, b) or $(b, x, c)=b$.

A simple four element example shows that the following cannot replace (O_3) in (11.2). Postulates M, N, and O_3 imply

(11.1) $$(a, b, c)=a, b, \text{ or } c.$$

PROOF. Let x in (O_3) be (a, b, c) and apply (8.4).

(11.2) Postulates M, N, and O_3 characterize chains.

PROOF. Assume (M), (N), and (O_3). Postulates B and D_3 follow by (8.9) and (11.1). By (7.1), we have a chain if Postulate C holds. Assume $abc\cdot bde$. If eba, (C) holds. If bea, we have in turn $bdea$, $cbdea$, cbd, and (C) holds. But if neither eba or bea holds, we have by (O_3) either bac or cae. In the first case, $bac\cdot abc\rightarrow a=b\rightarrow eba$, a contradiction. In the second case, we have in turn $eabc$, ebc, $edbc$, and dbc. Again, (C) holds. Conversely, since a chain is a tree it satisfies Postulates M and N. It remains to show (O_3) holds. Assume $(a, b, c)\neq a$ or c. By (D_3) we have abc. If abx, (O_3) follows. If not, $a\neq b$ and, from (D_3), either bxa or bax holds. In the first case, $abc\cdot bxa\rightarrow(cbx$ or $aba)\rightarrow cbx$. In the second case, $cba\cdot bax\cdot a\neq b\rightarrow cbax\rightarrow cbx$.

It is clear from the following that Postulates M, N, O_1, and O_2 characterize bounded chains.

(11.3) Postulates M, N, O_1, and O_2 imply Postulate O_3.

PROOF. Since $(a, 0, b)=((0, a, b), I, 0)=(0, a, I)$ or $(0, b, I)$, we have, say, $(a, 0, b)=a$. Then $(a, b, c)=((0, b, a), (0, b, b), c)=((a, b, c), 0, b)=(b, a, 0)$ or $(b, c, 0)$. Hence $(a, b, c)=a, b$, or c. If $(a, b, c)=b$, $b=(b, x, b)=((b, a, c), x, b)=(b, a, x)$ or (b, c, x) and (O_3) is established.

12. **Median latticoids.** It is interesting that trees and bounded distributive lattices have an algebra with many properties in common. Assume Postulates M and N hold for a set of elements S. Let 0 and I be elements arbitrarily chosen from S. Define $a+b$ as (a, I, b) and ab as $(a, 0, b)$. S is closed under these operations and

properties V and W are immediate. Properties X and Y follow respectively from (8.6) and (N).

(V) The idempotent laws hold.

(W) The commutative laws hold.

(X) The associative laws hold.

(Y) The distributive laws hold and hence $a+ab=a(a+b)$.

(Z) Elements 0 and I have the properties that, for all a and b in S, $a0=0$, $a+I=I$, and $a+0=Ia=I(a+ab)$.

PROOF. All properties except the last are immediate. The last follows from $(I, 0, (a, I, (a, 0, b)))=(I, 0, (a, b, (a, 0, I)))=((I, 0, a), b, (I, 0, (a, 0, I)))=((I, 0, a), b, (a, 0, I))=(I, 0, a)$.

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