In case K is a division ring, the set S coincides with the set generated by the perfect squares as used by Szele. This follows easily from the identity $xyx=(xy)^2(y^{-1})^2y$, $x, y \in K^*$.

If the domain of integrity K is ordered, say $K^*=P\cup(-P)$, then for an extension L of K the ordering of K can be extended to an ordering of L if and only if $T\subset L^*$, where T is the additive semigroup in L generated by P^E. The proof of this result is much the same as that of the above theorem. This generalizes Theorem 2 of Szele's paper to a domain of integrity.

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THE ZEROS OF AN ANALYTIC FUNCTION OF ARBITRARILY RAPID GROWTH[1]

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1. Introduction. It was shown by Poincaré [4],[2] Borel [1], and others that an integral function may be made "to grow" arbitrarily rapidly along the real axis or along other curves extending to infinity. Ketchum [2] has considered the corresponding problem for more general point sets. He investigated sets such that, for any given function $G(z)\geqq 0$, there exists a function $f(z)$ which is analytic except where $G(z)$ is unbounded and which satisfies the inequality

$$|f(z)| \geqq G(z)$$

for every point z of the set.

In the publication of his results Ketchum [2] proposed a corresponding problem in which the additional restriction is placed on the function $f(z)$ that it be nonvanishing except at certain specified points of the complement of the set. In particular, suppose $S_1, S_2, \cdots$ is an infinite sequence of simply-connected regions whose closures are nonintersecting and whose only "sequential limit point" is the point at infinity. Then, if $\{M_i\}$ is any preassigned sequence of positive constants, does there exist a nonvanishing integral function $f(z)$ such that $|f(z)|\geqq M_i$ when $z\in S_i$?

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[1] The author is greatly indebted to Professor P. W. Ketchum, who suggested this topic and gave valuable suggestions in its preparation.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

In the present paper it is proved that the above question can be answered in the affirmative, provided the $\bar{S}_i$'s do not separate the plane. Necessary and sufficient conditions are obtained in order that analogous results hold for more general point sets.

2. **Preliminary definitions.**[3] Throughout this paper the extended plane is to be understood.

A point which is a limit point of some set of points chosen one from each component of any given set S is called a *sequential limit point of* S (S-s.l. point).

A set S whose components are closed and whose s.l. points are in $C(S)$ (the complement of S) will be called a Q-set. We note that a Q-set has at most a denumerable number of components.

If S is a Q-set, any closed set E in $C(S)$ which contains the S-s.l. points will be called an E_S*-set.*

A set consisting of (1) the set B of the s.l. points of a given set S and (2) precisely one point of each component $I_k(S)$ of $C(S)$ such that $I_k(S) \cap B = \emptyset$ will be called a $B^*(S)$*-set.*

Suppose S is a Q-set and B is the set of S-s.l. points. Then a function $M(z)$ will be said to be A*-bounded on* S if there exists a function $g(z)$ such that:

(1) $|g(z)| > |M(z)|$ when $z \in S$,

(2) $g(z)$ is analytic in $C(B)$.

Let E denote an E_S-set. Then $M(z)$ is said to be $A^*(E)$*-bounded on* S if there exists a function $g(z)$ such that:

(1) $|g(z)| > |M(z)|$ when $z \in S$,

(2) $g(z)$ is analytic and nonvanishing in $C(E)$.

3. **Rate of growth theorems.** It is to be understood that all functions considered are single-valued. A function is said to be *analytic on a set* if it is analytic in a neighborhood of every point of the set.

Ketchum [2] has proved that every function which is bounded on each component of a Q-set S is A-bounded on S. In the present paper it is proved that a function which is component-wise bounded on a Q-set S whose complement is connected is $A^*(E)$-bounded on S, where E may be chosen to consist of just the set of S-s.l. points. More generally, if every component of $C(S)$ contains a point of E, any function which is component-wise bounded on S is $A^*(E)$-bounded on S. Theorem 2 gives a necessary and sufficient condition on the set E that a component-wise bounded function be $A^*(E)$-bounded.

[3] For examples, see [5].

THEOREM 1. *Suppose S is a Q-set and that B^* is any $B^*(S)$-set. Let $\{M_i\}$ and $\{N_i\}$ be arbitrary sequences of non-negative constants such that $M_i > N_i$. Then there exists a function $g(z)$ which satisfies the following conditions*: (1) *$g(z)$ is analytic and nonvanishing in $C(B^*)$*; (2) *for every i, $N_i < |g(z)| < M_i$ when $z \in S_i$ (where the components of S are $S_1, S_2, \cdots$).*

PROOF. Define ϵ_i and m_i thus:

$$\epsilon_i = \frac{(M_i - N_i)}{4}, \qquad m_i = \frac{M_i + N_i}{2}.$$

According to an approximation theorem of the author [5], there exists a function $f(z)$ nonvanishing and analytic in $C(B^*)$ such that, for every i,

$$|f(z) - m_i| < \epsilon_i \qquad \text{when } z \in S_i.$$

Hence, when $z \in S_i$,

$$|f(z)| < m_i + \epsilon_i < M_i$$

and

$$|f(z)| > m_i - \epsilon_i > N_i.$$

We take $f(z)$ for the required function $g(z)$.

The following topological theorems are stated for reference.

(A) *If E is any set and K is a connected set such that $K \cap E \neq \emptyset$ and $K \cap C(E) \neq \emptyset$, then $K \cap F(E) \neq \emptyset$* [3, p. 64] (where $F(E)$ denotes the boundary of E).

(B) *If S is a Q-set and B is its set of* s.l. *points, any component $I(S)$ of $C(S)$ such that $I(S) \cap B = \emptyset$ is a region of finite connectivity and $F\{I(S)\} \subset S$* [5].

(C) *If two points are separated by a closed set F, they are separated by a component of F* [3, p. 117].

Before proceeding to Theorem 2 we introduce some further topological definitions.

An infinite subset $\{S_{n_i}\}$ of components of a set S such that S_{n_i} is not separated from $S_{n_{i+1}}$ by any other component of S will be called *nested* if it can be arranged in an order $S_{n_1}, S_{n_2}, \cdots$ so that S_{n_i} separates S_{n_j} from S_{n_k} when $j < i < k$. An equivalent definition is that, when properly ordered, S_{n_j} separates S_{n_k} from the set of s.l. points of this sequence when $j > k$.

Suppose that S is a Q-set having an infinite nested sequence of components $\{S_{n_i}\}$. Then $\{S_{n_i}\}$ will be said to be *E-free* if there exists

N such that, for $i>N$, no point of E belongs to the component of $C(S)$ connecting S_{n_i} and $S_{n_{i+1}}$, i.e., if every point of E between S_{n_i} and $S_{n_{i+1}}$ is separated from S_{n_i} (or $S_{n_{i+1}}$) by a component of S.

If S is a Q-set and E is an E_S-set, the set S^* defined in the following manner will be called its $S^*(E)$*-set*. Let $S^*=S\cup_{k\in\psi}(I_k(S))$, where $k\in\psi$ if and only if $I_k(S)\cap E=\emptyset$. We note that if any point of S belongs to a component S_j^* of S^*, the component of S to which it belongs is entirely contained in that S_j^*.

LEMMA. *Let S be a Q-set and E an E_S-set such that S has no E-free infinite nested sequence of components. Then the following conditions are satisfied by S^*, the $S^*(E)$-set:*

(a) *For no component S_k^* of S^* is $S_k^*\cap S=\emptyset$.*

(b) *S^* is a Q-set having the same* s.l. *points as S.*

(c) *E contains a $B^*(S^*)$-set.*

PROOF OF (a). From the definition of S^*, either $S_k^*\cap S=\emptyset$ or there exists an integer t such that $S_k^*\cap I_t(S)\neq\emptyset$. In the latter case (B) implies $F\{I_t(S)\}\subset S$ and so $F\{I_t(S)\}\subset S_k^*$. We conclude that, in any case, $S_k^*\cap S\neq\emptyset$.

PROOF OF (b). It is sufficient to show that an arbitrary component S_k^* of S^* is a closed set and that the S^*-s.l. points are identical with the S-s.l. points, which lie in $C(S^*)$.

Let b denote an arbitrary point of $(\bar{S}_k^*-S_k^*)$. By arriving at a contradiction we shall prove that $(\bar{S}_k^*-S_k^*)$ is the null set.

Suppose S_t is any component of S such that $S_t\subset S_k^*$. We shall show that S_t is separated from b by an S_i—$S_{n_1}\subset S_k^*$, $i\neq t$; then by the same argument S_{n_1} is separated from b by some other component of S—$S_{n_2}\subset S_k^*$; etc. At each step we can choose the first component of S separating $S_{n_{i-1}}$ from b. Thus, an infinite nested sequence $\{S_{n_i}\}$ can be obtained. Moreover, this sequence is E-free, since it lies in S_k^*. But, according to the hypothesis, such a sequence does not exist. We then conclude that $(\bar{S}_k^*-S_k^*)$ is the null set, i.e., that S_k^* is a closed set.

In the above argument it remains yet to show that an arbitrary component $S_t\subset S_k^*$ is separated from b by an $S_i\subset S_k^*$, $i\neq t$.

Since $b\notin S_t(\subset S_k^*)$, $b\in I_n(S_t)$—some component of $C(S_t)$. Let q be a point of $\{I_n(S_t)\cap S_k^*\cap C(S)\}$ such that q is neither separated from S_t by any S_i nor from b by S_t. (To show the existence of such a point q first note that b—a limit point of S_k^*—lies in $I_n(S_t)$, which is a region, and then note that some point of $F\{I_n(S_t)\}$ $(\subset S_t)$ is a limit point of $\{S_k^*\cap I_n(S_t)\}$).

For some component $I_q(S)$ of $C(S)$, $q\in I_q(S)$. If $b\in I_q(S)$, $b\subset S_k^*$, contrary to definition of b. We conclude that $b\notin I_q(S)$. Then, by

(A) and (C), we know that q is separated from b by a component of $F\{I_q(S)\}$, and so by an S_i, say S_{n_1} (since (B) implies $F\{I_q(S)\}\subset S$). Now, since no S_i separates S_t from q and since S_t does not separate q and b, S_{n_1} separates S_t from b. Clearly, $S_{n_1}\subset S_k^*$, since $I_q(S)\subset S_k^*$ and so $F\{I_q(S)\}\subset S_k^*$.

By the previous argument, this completes the proof that S_k^* is closed.

We proceed to the proof that the S^*- and S-s.l. points are identical. We first note that each S_j^* contains only a finite number of S_i's. For it is apparent from the definition of S^* that any S-s.l. point lies in $C(S^*)$ and it was just shown that S_j^* is closed.

If b is any S-s.l. point, every neighborhood of b contains points of an infinity of S_i's, hence—by the above observations—points of an infinity of S_j^*'s. That is, b is an S^*-s.l. point.

On the other hand, if b^* is an S^*-s.l. point, it can easily be verified that b^* is an S-s.l. point.

Proof of (c). To verify that E contains a $B^*(S^*)$-set, it remains only to be shown that every component of $C(S^*)$ contains a point of E. This is a consequence of the definition of S^* and of the fact that any component of $C(S^*)$ is just a component of $C(S)$.

Before stating Theorem 2 we define a Q_R-set. A Q-set S will be called a Q_R-set if for every component $I_j(S_t)$ of $C(S_t)$ (where S_t is any component of S) the following condition is satisfied: $\bar{I}_j(S_t)$ does not separate the plane. We note that any Q-set whose components are closed regions is a Q_R-set. An example of a Q-set which is not a Q_R-set is given at the end of this paper.

Theorem 2. *Let S be a Q_R-set and E an E_S-set. Then a necessary and sufficient condition that every function $M(z)$ which is component-wise bounded on S be $A^*(E)$-bounded on S is that S have no E-free infinite nested sequence of components.*

Proof. According to the lemma, S^*, the $S^*(E)$-set, is a Q-set and E contains a $B^*(S^*)$-set. It was previously noted that each S_j^* contains only a finite number of S_i's.

Let $M_i=\text{l.u.b. } |M(z)|$ on S_i.

Let $N_j^*=\text{maximum } M_i$ for the S_i's such that $S_i\subset S_j^*$, and let $M_j^*=N_j^*+1/4$.

Theorem 1 shows the existence of a function $g(z)$ analytic and non-vanishing in $C(E)$ such that, when $z\in S_j^*$,

$$N_j^* < |g(z)| < M_j^*.$$

Now $|g(z)| > |M(z)|$ on S. This completes the proof of the sufficiency of the condition.

Let us proceed to the proof of the converse. The argument is by contradiction; that is, it is assumed that S *does* have an E-free infinite nested sequence of components. A theorem of Ketchum is applied which shows the existence of a function $G(z)$ which is not A-bounded in a certain region R. Then a function $f(z)$ is defined on S such that $|f(z)| > |G(z)|$ on S. Finally, by applying the minimum modulus theorem we shall show that $|f(z)| > |G(z)|$ everywhere in R—contrary to the definition of $G(z)$. This leads to the conclusion that S has no E-free infinite nested sequence of components.

Suppose that S *does* have an E-free infinite nested sequence of components $\{S_{n_i}\}$ whose set of s.l. points we designate by $B^{(i)}$. Then there exists k such that, when $i \geqq k$, no point of E belongs to the component of $C(S)$ connecting S_{n_i} and $S_{n_{i+1}}$. We let $O(S_{n_k})$ designate the component of $C(S_{n_k})$ which contains $B^{(i)}$.

Now let S' denote the subset of S which consists of those points of the S_{n_i}'s in $O(S_{n_k})$ and of those points of components of S which lie between some S_{n_i} and $S_{n_{i+1}}$, for $i > k$, and which separate the plane but which are not separated from S_{n_i} (or $S_{n_{i+1}}$) by a component of S.

Next a certain region R is defined in which Ketchum's theorem is applicable. Let $I_j(S_t')$ denote a component of $C(S_t')$ (where S_t' is a component of S'). We shall say that $j \in \psi_t$ if

(a) $I_j(S_t') \subset O(S_{n_k})$, and

(b) $I_j(S_t') \cap E \neq \emptyset$ but $I_j(S_t') \cap B^{(i)} = \emptyset$.

Let

$$C_1 = \bigcup_{j \in \psi_1} \bar{I}_j(S_1'),$$
$$\cdots\cdots\cdots,$$
$$C_m = \bigcup_{j \in \psi_m} \bar{I}_j(S_m'),$$
$$\cdots\cdots\cdots,$$

and let

$$F = \{\bigcup_m C_m \cup B^{(i)}\}.$$

Now define

$$R = \{I_n(B) \cap O(S_{n_k})\} - F,$$

where $I_n(B)$ designates the component of $C(B)$ which contains S'.

We note that F is a closed set and that R is open. Furthermore,

R is connected. For, if some two points x and y belonged to different components of R, they would be separated by $F\{R\}\subset(B\cup F\cup S_{n_k})$—hence, either by a component of $(F\cup B)$ or by S_{n_k}. (See (A) and (C).) Since $x, y\in I_n(B)$, they are not separated by B. If x and y were separated by some other component of F, they would be separated by some $\bar{I}_j(S'_t)$, contrary to the definition of S as a Q_R-set (since S'_t is just a component of S). Certainly, S_{n_k} does not separate x and y. We conclude that R is a region.

A theorem of Ketchum [2, Theorem 5] shows the existence of a function $G(z)$ such that (a) $B^{(i)}$ is identical with the set of points in every neighborhood of which $G(z)$ is unbounded and (b) there exists no function $f(z)$ meromorphic in R such that $|f(z)|\geqq|G(z)|$ everywhere in R. Let us choose such a function $G(z)$.

We now show the existence of a function $f(z)$ analytic in R such that $|f(z)|>|G(z)|$ everywhere in R, thus obtaining a contradiction and showing that the assumption "*E does* have an E-free infinite nested sequence of components" is false.

Let W_t denote the set of points of R which lie between $S_{n_{t-2}}$ and $S_{n_{t+2}}$. Let

$$\begin{aligned} M_k &= \text{l.u.b. } |G(z)| \text{ on } W_k;\\ M_{k+1} &= \text{l.u.b. } |G(z)| \text{ on } W_{k+1};\\ &\cdots\cdots\cdots\cdots\cdots;\\ M_{k+j} &= \text{l.u.b. } |G(z)| \text{ on } W_{k+j};\\ &\cdots\cdots\cdots\cdots\cdots. \end{aligned}$$

Now define

$$\begin{aligned} M(z) &= M_k \text{ on } S_{n_k}\\ &= M_{k+j} \text{ on } S_{n_{k+j}}\\ &= \max(M_k, M_{k+1}) \quad \text{on those components of } S \text{ between } S_{n_k} \text{ and } S_{n_{k+1}}\\ &= 1 \quad \text{on all other components of } S. \end{aligned}$$

By hypothesis, every function which is component-wise bounded on S is $A^*(E)$-bounded; hence, there exists $f(z)$ nonvanishing and analytic in $C(E)$ and such that

$$|f(z)| > M(z) \text{ on } S.$$

It is next shown that $f(z)$, which is analytic in R, satisfies the inequality

(1) $$|f(z)| > |G(z)| \qquad \text{everywhere in } R,$$

contrary to the choice of $G(z)$.

Let p be any point of R. Clearly, (1) is satisfied if $p \in S'$. If $p \notin S'$, $p \in I_j(S')$ for some component $I_j(S')$ of $C(S')$. Either

(i) $F\{I_j(S')\} \subset S_{n_t}$ for some t

or

(ii) $I_j(S')$ lies between S_{n_t} and $S_{n_{t+1}}$, for some t, and $F\{I_j(S')\}$ is contained in S_{n_t}, $S_{n_{t+1}}$, and in components of S' between S_{n_t} and $S_{n_{t+1}}$.

In either case, $I_j(S') \subset \{W_t \cap W_{t+1}\}$.

In case (i), $I_j(S')$ is just a component of $C(S_{n_t})$, which was not omitted in forming R, and so $I_j(S') \cap E = \emptyset$. In case (ii), it follows from the definition of S_{n_k} and of F that $I_j(S') \cap E = \emptyset$.

Hence, $f(z)$ is nonvanishing in $I_j(S')$ and the minimum modulus theorem implies

(2) $$|f(p)| \geqq \min_{\text{on } F\{I_j(S')\}} |f(z)|.$$

When z is a point of S_{n_t}, of S_{n_t+1}, or any point of S' between these two components,

$$|f(z)| > M(z) \geqq \min (M_t, M_{t+1}) \geqq \operatorname*{l.u.b.}_{\text{on } W_t \cap W_{t+1}} |G(z)|.$$

Hence, on $F\{I_j(S')\}$,

(3) $$|f(z)| > \text{l.u.b. } |G(z)| \text{ on } W_t \cap W_{t+1}.$$

Combining (2) and (3), we obtain

$$|f(p)| > \operatorname*{l.u.b.}_{\text{on } I_j(S')} |G(z)| \geqq |G(p)|.$$

This completes the proof that $|f(z)| > |G(z)|$ in R, contrary to the definition of $G(z)$, and thus shows the necessity of the condition that S have no E-free infinite nested sequence of components.

The following example seems to indicate that the condition in Theorem 2 that S be E-free might not be a necessary condition if S were required to be just a Q-set, instead of a Q_R-set. In the proof of the sufficiency, the assumption that S be a Q-set would have sufficed.

Given an infinite sequence of concentric circular rings with centers at the origin, having infinity as the only s.l.-point, let S be the point set obtained by joining the 1st and 2nd by a line segment, say a segment of the real axis, the 3rd and 4th by such a segment, and, in general, the $(2n-1)$st and the $2n$th. The components of $C(S)$ are of two types: (1) those regions connecting the components of S, which are just circular rings, and (2) those regions which are circular rings with a cross segment deleted.

It follows readily from a theorem proved by Ketchum [2] that, for any function $G(z)$ bounded on each component of S, there is an integral function $f(z)$ such that $|f(z)| > |G(z)|$ on S, and any zeros of $f(z)$ may be required to lie in regions of type (2).[4] That is, the set E of zeros is such that S is E-free. Of course, since the exact location of the zeros was not preassigned, this merely shows that all those regions connecting S_i and S_{i+1}, $i=1, 2, \cdots$, may be required to be free of zeros.

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[4] Fill in the regions of type (1) and apply Ketchum's Theorem VI on the set obtained.