and add the members of the resulting equations; we then have

$$A(X_0 + X_1B + \cdots + X_pB^p) - (X_0 + X_1B + \cdots + X_pB^p)B = C.$$

Hence $X_0+X_1B+ \cdots +X_pB^p$ with elements in F is a solution of (1) where p is obviously less than n. Similarly $Y_0+AY_1+ \cdots +A^pY_p$ is also a solution. The theorem is proved.[2]

UNIVERSITY OF TULSA

[2] Professor Jacobson, Editor of these Proceedings, in a letter to me dated June 15, 1951, says in part: The referee "noted that the second theorem can be formulated in terms of groups with operators from a principal ideal domain." And he expressed the opinion that a formulation in finite abelian groups may serve the same purpose. Subsequently, the writer has read, perhaps with insufficient understanding, Chatelet's dissertation, University of Paris, 1911, and has *not* found therein the group theoretic formulation noted by the referee as resulting from Theorem II above, and hopes the referee's observation is new.

SYMMETRY OF BANACH ALGEBRAS[1]

IRVING KAPLANSKY

1. **Introduction.** Gelfand and Neumark [2][2] raised the question as to whether a B^*-algebra is necessarily symmetric.[3] This question remains open. Also open, of course, is the more general question propounded by the author in [3]: is a C-symmetric algebra symmetric? The latter question is of some independent interest, since an affirmative answer would ease the task of proving that a specific algebra (such as a group algebra) is symmetric. Consequently we shall devote this note to pushing forward the second question to the same point that the first has reached. We shall prove the following generalization of [4, Lemma 7.2].

THEOREM. *Let A be a C-symmetric algebra with continuous involution. Then for no element x in A can $-x^*x$ be a nonzero idempotent.*

The proof rests largely on a purely algebraic result (Lemma 5) which appears to be new even in the finite-dimensional case.

2. **Definitions.** Our Banach algebras will admit complex scalars,

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[1] The preparation of this paper was sponsored in part by the Office of Naval Research.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

[3] For the relevant definitions, see §3.

although everything can be readily extended to the real case. A Banach algebra is a *-algebra if it possesses an involution * (conjugate linear anti-automorphism of period two). A *-algebra A is *symmetric* if x^*x is quasi-regular for all x in A; it is C-symmetric if every closed commutative *-subalgebra is symmetric. The spectrum $S(x)$ of an element x in a Banach algebra is the set of all scalars λ such that $-\lambda^{-1}x$ is not quasi-regular; it is a matter of indifference how we treat the insertion of 0 in the spectrum. We quote as explicit lemmas some facts which are proved in [**3**].

LEMMA 1. *For any element x in a symmetric algebra, $S(x^*x) \geqq 0$.*

LEMMA 2. *For any element x in a C-symmetric algebra, $x^* - x$ is quasi-regular.*

LEMMA 3. *A C-symmetric algebra satisfying a polynomial identity is symmetric.*

The following result was proved in [**4**, Lemma 7.1] on the hypothesis that $y^*y = 0$ implies $y = 0$. We now show that C-symmetry is also an adequate assumption.

LEMMA 4. *Let A be a C-symmetric algebra with unit element. Then for no x in A can x^*x be -1.*

PROOF. Suppose that $x^*x = -1$. Write $e = 1 + xx^*$, $z = xe$. One verifies that $e^2 = -z^*z = e$, $ez = z^2 = 0$, whence

$$(1 + z^* - z)(e + z) = 0.$$

By Lemma 2, $1 + z^* - z$ is regular. Hence $e + z = 0$, and $e = 0$. But $x^*x = xx^* = -1$ is impossible in a C-symmetric algebra.

3. **An algebraic lemma.** Let F be a field, and F_2 the two by two matrix algebra over F. It is proved in [**1**] that F_2 satisfies the identity

$$\sum \pm x_1x_2x_3x_4 = 0, \tag{1}$$

where the sum is taken over all permutations, and the sign is determined by the parity of the permutation. Moreover, for $n > 2$, the n by n matrix algebra over F does not satisfy (1).

LEMMA 5. *Let F be a field, and A the free algebra (with coefficients in F) generated by x and y subject to the relations $x^2 = \alpha x$, $y^2 = \beta y$, where α and β are in F. Then A satisfies* (1).

REMARKS. 1. If $\alpha \neq 0$ we can normalize by taking $\alpha^{-1}x$ instead of x, etc. In this way we see that there are just three cases covered by the lemma: two idempotents, or two elements with square 0, or one of each.

2. Let us note the following explicit special case of Lemma 5: Let e and f be idempotents in a finite-dimensional algebra, B the subalgebra generated by e and f, and R the radical of B. Then B/R is a direct sum of fields and two by two matrix algebras over fields.

PROOF. Since (1) is multilinear, it is enough to verify it for a basis of A. We get a suitable basis for A by taking the monomials built out of x and y; in view of the relations $x^2=\alpha x$, $y^2=\beta y$, we need only consider products which alternate between x and y. While there are an infinite number of these monomials, they fall into four types:

$$\text{(a)}\quad x\cdots x,\qquad \text{(b)}\quad x\cdots y,\qquad \text{(c)}\quad y\cdots x,\qquad \text{(d)}\quad y\cdots y.$$

Now if on substituting in (1) we use two elements of the same type, it is clear that the resulting expression is invariant under interchange of the two monomials in question. On the other hand, (1) changes sign under a transposition. Hence such a substitution makes (1) vanish.[4]

There remains the case where we substitute 4 different types. This case can be settled by direct computation. Of the 24 products, precisely 6 turn out to be of each of the 4 types. By symmetry, we need only examine the products that wind up as types (a) or (b). Those of type (a) are *abdc*, *adbc*, *badc*, *bcda*, *bdac*, *bdca*; and they yield the same monomial with coefficients $-\alpha\beta^2$, β, β, $-\beta$, $-\beta$, $\alpha\beta^2$, adding up to 0. Similarly the products of type (b) are *abcd*, *acbd*, *acdb*, *adcb*, *bacd*, *bcad*; the coefficients are $\alpha\beta$, $-\alpha\beta$, 1, $-\alpha\beta$, -1, $\alpha\beta$, and again add up to 0. This completes the proof of Lemma 5.

4. **Proof of the theorem.** Suppose that $x^*x=-e$, where e is a (self-adjoint) idempotent. We shall prove $e=0$.

Since we can replace x by xe if necessary, there is no loss of generality in supposing that $x=xe$. Write $f=-xx^*$; then f is an idempotent. Write further $y=ex$, $z=x-ex$. We have

$$z^*z = -(e+y^*y). \tag{2}$$

Now $z^2=(z^*)^2=0$. By Lemma 5, the closed subalgebra generated by z and z^* satisfies the identity (1). Since $*$ is continuous, B is self-adjoint, hence C-symmetric. It follows (Lemma 3) that B is actually symmetric, and hence (Lemma 1) $S(z^*z)\geqq 0$.

Next we turn our attention to the element y^*y, and shall prove that

[4] This argument can be patched up for characteristic 2 by a standard device.

$-1 \leqq S(y^*y) \leqq 0$. It is known that y^*y and yy^* have identical spectra,[5] except possibly at 0; so it is equivalent to prove that $yy^* = -efe$ has spectrum in the interval $(-1, 0)$. The closed subalgebra generated by e and f is self-adjoint, satisfies (1) by Lemma 5, hence is symmetric by Lemma 3. Since

$$efe = (fe)^*(fe), \qquad e - efe = (e - fe)^*(e - fe),$$

we deduce from Lemma 1 that $S(efe) \geqq 0$, $S(e-efe) \geqq 0$. This shows that $S(efe)$ lies in the interval $(0, 1)$. We have proved the desired fact that $S(y^*y)$ lies in $(-1, 0)$. This in turn implies $S(e+y^*y) \geqq 0$, since e is a unit element for y^*y. In the light of equation (2), we have $S(z^*z) \leqq 0$. But we showed above that $S(z^*z) \geqq 0$. Hence $S(z^*z) = 0$, that is

$$\lim \left\| (z^*z)^n \right\|^{1/n} = 0. \tag{3}$$

From (3) it is easy to see that the binomial expansion of $(e+z^*z)^{-1/2}$ converges say to w. Then

$$w(e + z^*z)w = e. \tag{4}$$

By the continuity of $*$, we have $w^* = w$. On combining (2) and (4) we find

$$wy^*yw = (yw)^*(yw) = -e.$$

Since e is a two-sided unit element for yw, we have contradicted Lemma 4, unless $e = 0$.

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University of Chicago

[5] This follows from the fact (valid in any ring) that xy is quasi-regular if and only if yx is quasi-regular; cf. Jacobson, Amer. J. Math. vol. 67 (1945) p. 302.