RIEMANN'S METHOD AND THE PROBLEM OF CAUCHY. II. THE WAVE EQUATION IN n DIMENSIONS[1]

J. B. DIAZ AND M. H. MARTIN

1. Introduction. In a recent paper[2] Riemann's method for the solution of the problem of Cauchy for a linear hyperbolic partial differential equation $L(u)=0$ of second order for one unknown function u of two independent variables x, y was modified by the introduction of a line integral $I_1=\int\{Bdx-Ady\}$ vanishing on closed paths. Here A and B are bilinear forms in the partial derivatives u_x, u_y, v_x, v_y; and v, the ***resolvent***, is a properly chosen solution (analogous to Riemann's function) of an associate equation $M(v)=0$, the counterpart to the adjoint equation.

This modification opened the way to an extension of Riemann's method to the wave equation

$$u_{xx} + u_{yy} - u_{tt} = 0,$$

in two dimensions. The line integral I_1 was replaced by an integral I_2 vanishing on closed surfaces and the associate equation $M(v)=0$ turned out to be the Euler-Poisson equation[3]

$$M(v) = v_{\alpha\beta} + \frac{1/2}{\alpha - \beta}(v_\alpha - v_\beta) = 0,$$

with the resolvent

$$v = \alpha + \beta + 2[(\bar{t} - \alpha)(\bar{t} - \beta)]^{1/2}$$

taking over the role of Riemann's function.

In the present paper the authors extend this method to the wave equation

$$u_{x_1x_1} + \cdots + u_{x_nx_n} - u_{tt} = 0,$$

in n dimensions, $n \geqq 2$, with, as might be expected, an n-dimensional integral I_n, which vanishes over closed n-dimensional surfaces bounding $(n+1)$-dimensional volumes, replacing I_1 and I_2. The associate

Presented to the Society, December 27, 1951; received by the editors November 19, 1951.

[1] This paper was sponsored by the Office of Naval Research.

[2] M. H. Martin, *Riemann's method and the problem of Cauchy*, Bull. Amer. Math. Soc. vol. 57 (1951) pp. 238–249.

[3] G. Darboux, *Leçons sur la théorie générale des surfaces*, 2d ed., vol. II, p. 54 ff., Paris, 1914–1915.

equation is now

$$M(v) = v_{\alpha\beta} + \frac{(n-1)/2}{\alpha - \beta}(v_\alpha - v_\beta) = 0,$$

and the resolvent is

$$v = (\bar{t} - \alpha)^{(n-1)/2}(\bar{t} - \beta)^{(n-1)/2}.$$

2. **The Laplacian $\Delta_2 u = u_{x_1x_1} + \cdots + u_{x_nx_n}$ in polar coordinates.** Consider the generalization to n dimensions of the well known space polar coordinate system ϕ, θ, r, in three dimensions, where

$$x = r\cos\phi\sin\theta, \qquad y = r\sin\phi\sin\theta, \qquad z = r\cos\theta,$$
$$0 \leqq \phi < 2\pi,\ 0 \leqq \theta \leqq \pi,\ r \geqq 0,$$

that is, coordinates $\phi, \theta_1, \cdots, \theta_{n-2}, r$ with

$$
\begin{aligned}
x_1 &= r\cos\phi\sin\theta_1\sin\theta_2\sin\theta_3\cdots\sin\theta_{n-2}, && 0 \leqq \phi < 2\pi,\\
x_2 &= r\sin\phi\sin\theta_1\sin\theta_2\sin\theta_3\cdots\sin\theta_{n-2}, && 0 \leqq \theta_1 \leqq \pi,\\
x_3 &= r\cos\theta_1\sin\theta_2\sin\theta_3\cdots\sin\theta_{n-2}, && 0 \leqq \theta_2 \leqq \pi,\\
x_4 &= r\cos\theta_2\sin\theta_3\cdots\sin\theta_{n-2}, && 0 \leqq \theta_3 \leqq \pi,\\
&\cdots\cdots\cdots\cdots\cdots, && \cdot\\
x_{n-1} &= r\cos\theta_{n-3}\sin\theta_{n-2}, && 0 \leqq \theta_{n-2} \leqq \pi,\\
x_n &= r\cos\theta_{n-2}, && r \geqq 0.
\end{aligned}
\tag{1}
$$

The element of arc is given by

$$ds^2 = r^2\sin^2\theta_1\cdots\sin^2\theta_{n-2}d\phi^2 + r^2\sin^2\theta_2\cdots\sin^2\theta_{n-2}d\theta_1^2 + \cdots$$
$$+ r^2 d\theta_{n-2}^2 + dr^2,$$

and if we write

$$y_1 = \phi,\ y_2 = \theta_1,\ \cdots,\ y_{n-1} = \theta_{n-2},\ y_n = r,$$
$$g_{11} = r^2\sin^2\theta_1\cdots\sin^2\theta_{n-2},$$
$$g_{22} = r^2\sin^2\theta_2\cdots\sin^2\theta_{n-2},\ \cdots,\ g_{n-1,n-1} = r^2, \qquad g_{nn} = 1.$$

we shall have

$$\Delta_2 u = \frac{1}{g^{1/2}}\sum_{i=1}^{n}\frac{\partial}{\partial y_i}\left(\frac{g^{1/2}}{g_{ii}}u_{y_i}\right),\ g^{1/2} = r^{n-1}\sin\theta_1\sin^2\theta_2\cdots\sin^{n-2}\theta_{n-2}.$$

If we set $f_{i-1} = g^{1/2}/r^{n-3}g_{ii}$, $i = 1, \cdots, n-1$, so that

$$
\begin{aligned}
f_0 &= \csc\theta_1 \sin\theta_3 \sin^2\theta_4 \cdots \sin^{n-4}\theta_{n-2};\\
f_1 &= \sin\theta_1 \sin\theta_3 \sin^2\theta_4 \cdots \sin^{n-4}\theta_{n-2};\\
f_2 &= \sin\theta_1 \sin^2\theta_2 \sin\theta_3 \sin^2\theta_4 \cdots \sin^{n-4}\theta_{n-2};\\
f_3 &= \sin\theta_1 \sin^2\theta_2 \sin^3\theta_3 \sin^2\theta_4 \sin^3\theta_5 \cdots \sin^{n-4}\theta_{n-2};\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots;\\
f_{n-3} &= \sin\theta_1 \sin^2\theta_2 \cdots \sin^{n-3}\theta_{n-3} \sin^{n-4}\theta_{n-2};\\
f_{n-2} &= \sin\theta_1 \sin^2\theta_2 \cdots \sin^{n-2}\theta_{n-2};
\end{aligned}
\tag{2}
$$

we find

$$\Delta_2 u = f^{-1}\left[\frac{\partial}{\partial\phi}\left(\frac{f_0}{r^2}\, u_\phi\right) + \sum_{j=1}^{n-2} \frac{\partial}{\partial\theta_j}\left(\frac{f_j}{r^2}\, u_{\theta j}\right)\right] + u_{rr} + \frac{n-1}{r}\, u_r,$$

provided we write

$$f = \sin\theta_1 \sin^2\theta_2 \cdots \sin^{n-2}\theta_{n-2} = f_{n-2}.$$

We note in passing that the element of $(n-1)$-dimensional area on the unit sphere $r=1$ is

$$d\omega_n = f d\phi d\theta_1 \cdots d\theta_{n-2},$$

and the $(n-1)$-dimensional area of the unit sphere is

$$\omega_n = \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} f d\phi d\theta_1 \cdots d\theta_{n-2} = \frac{2\pi^{n/2}}{\Gamma(n/2)}\cdot$$

3. **A fundamental identity.** Starting with the polar coordinates $\phi, \theta_1, \cdots, \theta_{n-2}, r$ in n-dimensional space we introduce coordinates $\alpha, \beta, \phi, \theta_1, \cdots, \theta_{n-2}$ in $(n+1)$-dimensional space-time by setting

$$\alpha = t + r, \qquad \beta = t - r, \tag{3}$$

and term α, β *characteristic coordinates*, inasmuch as $\alpha=$const., $\beta=$const. are characteristic half-cones for the wave equation

$$L(u) = \frac{f}{4}(u_{tt} - \Delta_2 u) = 0.$$

In these coordinates the operator $L(u)$ takes the form

$$
\begin{aligned}
L(u) &= \left[u_{\alpha\beta} - \frac{(n-1)/2}{\alpha-\beta}(u_\alpha - u_\beta)\right] f\\
&\quad - (\alpha-\beta)^{-2}\left[\frac{\partial}{\partial\phi}(f_0 u_\phi) + \sum_{j=1}^{n-2} \frac{\partial}{\partial\theta_j}(f_j u_{\theta_j})\right],
\end{aligned}
\tag{4}
$$

with which we associate the operator

$$M(v) = v_{\alpha\beta} + \frac{(n-1)/2}{\alpha - \beta}(v_\alpha - v_\beta).$$

If we write

$$A = fu_\beta v_\beta, \qquad B = -fu_\alpha v_\alpha,$$

$$\Phi = f_0 \frac{v_\alpha - v_\beta}{(\alpha-\beta)^2} u_\phi, \qquad \Theta_j = f_j \frac{v_\alpha - v_\beta}{(\alpha-\beta)^2} u_{\theta_j},$$

$j=1, \cdots, n-2$, a simple calculation shows that (note that $v = v(\alpha, \beta)$)

$$A_\alpha + B_\beta + \Phi_\phi + \sum_{j=1}^{n-2} \frac{\partial \Theta_j}{\partial \theta_j} = (v_\beta - v_\alpha)L(u) + (u_\beta - u_\alpha) f M(v). \tag{5}$$

This identity plays the role of a Green's identity[4] in our investigation, the part of the adjoint equation being taken over by the *associate equation* $M(v)=0$.

According to the generalized Green's theorem, the surface integral

$$\begin{aligned} I_n = \int_{S_n} \{ & A d\beta d\phi d\theta_1 \cdots d\theta_{n-2} + B d\alpha d\phi d\theta_1 \cdots d\theta_{n-2} \\ & + \Phi d\alpha d\beta d\theta_1 \cdots d\theta_{n-2} + \cdots + \Theta_{n-2} d\alpha d\beta d\phi \cdots d\theta_{n-3} \} \\ = \int_{S_n} \{ & A\nu_\alpha + B\nu_\beta + \Phi\nu_\phi + \cdots + \Theta_{n-2}\nu_{\theta_{n-2}} \} dS_n \end{aligned} \tag{6}$$

(where $\nu_\alpha, \nu_\beta, \cdots$ are the components of the unit outer normal to S_n) when extended around a closed n-dimensional surface S_n bounding an $(n+1)$-dimensional volume V_{n+1} can be expressed as a volume integral over V_{n+1}, namely

$$\int_{V_{n+1}} \left(A_\alpha + B_\beta + \Phi_\phi + \sum_{j=1}^{n-2} \frac{\partial \Theta_j}{\partial \theta_j} \right) d\alpha d\beta d\phi d\theta_1 \cdots d\theta_{n-2}.$$

The following lemma is now obvious.

LEMMA. *The surface integral I_n, taken around a closed n-dimensional surface S_n, vanishes whenever u, v are regular solutions of $L(u)=0$, and its associate equation $M(v)=0$, respectively.*

---

[4] Compare the "formule fondamentale" in the terminology of J. Hadamard, *Le problème de Cauchy et les équations aux derivées partielles linéaires hyperboliques*, Paris, 1932, chapter II, esp. p. 83.

It is worth while to note that each of A, B, Φ, $\Theta_1, \cdots, \Theta_{n-2}$ is a bilinear form in the partial derivatives of first order of u and v with respect to α, β, ϕ, $\theta_1, \cdots, \theta_{n-2}$.

4. The problem of Cauchy. As Cauchy data on the hyperplane $t=0$ in $(n+1)$-dimensional space-time we take

$$u(x_1, \cdots, x_n, 0) = u^0(x_1, \cdots, x_n),$$
$$u_t(x_1, \cdots, x_n, 0) = u^1(x_1, \cdots, x_n),$$

the functions u^0, u^1 being given in advance. Let $P_{\bar{t}}$ denote the point with coordinates $(\bar{x}_1, \cdots, \bar{x}_n, \bar{t})$ in space-time. The solution of the problem of Cauchy requires the value $u(P_{\bar{t}})$ of the solution u of $L(u)=0$ to be expressed in terms of the initial data u^0, u^1 carried by the part of the initial hyperplane $t=0$ contained within the ("retrograde") characteristic half-cone with vertex at $P_{\bar{t}}$, i.e., in terms of the initial data assigned to the points

$$(x_1 - \bar{x}_1)^2 + \cdots + (x_n - \bar{x}_n)^2 \leqq \bar{t}^2, \qquad t = 0.$$

We assume $\bar{t}>0$ and consider the $(n+1)$-dimensional conical volume C bounded in space-time by the characteristic hypercone with vertex at $P_{\bar{t}}$ and the initial hyperplane $t=0$. The axis of C is the straight line $P_0P_{\bar{t}}$ in space-time traced out by P_t as t ranges from 0 to $\bar{t}$. If at each point P_t we introduce polar coordinates $\phi, \theta_1, \cdots, \theta_{n-2}, r$ with pole at P_t, the conical volume C is described by the inequalities

$$C\colon\quad 0 \leqq \phi < 2\pi, \quad 0 \leqq \theta_j \leqq \pi, \quad 0 \leqq r \leqq \bar{t} - t, \quad 0 \leqq t \leqq \bar{t} \qquad (j = 1, \cdots, n-2).$$

When we take $\alpha, \beta, \phi, \theta_1, \cdots, \theta_{n-2}$ as rectangular coordinates in a second $(n+1)$-dimensional space, C appears as a "wedge"

$$W\colon\quad 0 \leqq \alpha \leqq \bar{t}, \quad -\alpha \leqq \beta \leqq +\alpha, \quad 0 \leqq \phi < 2\pi, \quad 0 \leqq \theta_j \leqq \pi \qquad (j = 1, \cdots, n-2).$$

That part of the boundary of C formed by the mantle of the characteristic hypercone becomes the face $\alpha=\bar{t}$ of W; the base $t=0$ of C is represented by the face $\beta=-\alpha$ of W; and the axis $P_0P_{\bar{t}}$ of C by the face $\beta=\alpha$ of W. The vertex $P_{\bar{t}}$ of C appears as the edge $\alpha=\beta=\bar{t}$ of W; the periphery of the base of C (the intersection of the initial plane with the characteristic hypercone) is replaced by the edge $\alpha=-\beta=\bar{t}$ of W; and center P_0 of the base of C by the edge[5] $\alpha=\beta=0$ of W.

To reformulate the problem of Cauchy in $(\alpha, \beta, \phi, \theta_1, \cdots, \theta_{n-2})$-

[5] Compare M. H. Martin, loc. cit., p. 245.

space we observe that the carrier $t=0$ becomes the hyperplane $\beta=-\alpha$ upon which, from (3), we assign

$$(7) \quad u_\phi = \overset{0}{u}_\phi, \quad u_{\theta_j} = \overset{0}{u}_{\theta_j}, \quad u_\alpha = (\overset{0}{u}_r + \overset{0}{u}_t)/2, \quad u_\beta = -(\overset{0}{u}_r - \overset{0}{u}_t)/2$$

as initial data. One would accordingly seek an expression for the value of the solution u of $L(u)=0$, for $L(u)$ as defined in (4), along the edge $\alpha=\beta=\bar{t}$ of W in terms of the above initial data carried by the face $\beta=-\alpha$ of W.

To solve the problem of Cauchy as originally formulated we apply the lemma of the preceding section to the closed surface S_n which is the boundary of the wedge W and obtain

$$\underset{\beta=\alpha}{I_n} + \underset{\beta=-\alpha}{I_n} + \underset{\alpha=\bar{t}}{I_n} + \left(\underset{\phi=0}{I_n} + \underset{\phi=2\pi}{I_n}\right) + \sum_{j=1}^{n-2}\left(\underset{\theta_j=0}{I_n} + \underset{\theta_j=\pi}{I_n}\right) = 0.$$

For single-valued solutions, u must be periodic of period 2π in ϕ and it follows from the definition of Φ that

$$\underset{\phi=0}{I_n} + \underset{\phi=2\pi}{I_n} = 0,$$

since the external normals to S_n have opposite directions on the faces $\phi=0$, $\phi=2\pi$. Since Θ_j involves f_j, and f_j contains $\sin\theta_j$ as a factor for $j=1, \cdots, n-2$, it is clear that

$$\underset{\theta_j=0}{I_n} = \underset{\theta_j=\pi}{I_n} = 0,$$

and the above result simplifies to

$$\underset{\beta=\alpha}{I_n} + \underset{\beta=-\alpha}{I_n} + \underset{\alpha=\bar{t}}{I_n} = 0.$$

The integration of I_n in (6) over S_n yields

$$\int_0^{\bar{t}}\int_{\omega_n}[-A+B]_{\beta=\alpha}f^{-1}d\omega_n d\alpha - \int_0^{\bar{t}}\int_{\omega_n}[A+B]_{\beta=-\alpha}f^{-1}d\omega_n d\alpha$$
$$+ \int_{-\bar{t}}^{\bar{t}}\int_{\omega_n} A\Big]_{\alpha=\bar{t}} f^{-1}d\omega_n d\beta = 0,$$

and when we employ the definitions of A and B, we find

$$(8) \quad \begin{aligned} &-\int_0^{\bar{t}}\int_{\omega_n}[u_\alpha v_\alpha + u_\beta v_\beta]_{\beta=\alpha}d\omega_n d\alpha \\ &+\int_0^{\bar{t}}\int_{\omega_n}[u_\alpha v_\alpha - u_\beta v_\beta]_{\beta=-\alpha}d\omega_n d\alpha + \int_{-\bar{t}}^{\bar{t}}\int_{\omega_n} u_\beta v_\beta \int_{\alpha=\bar{t}} d\omega_n d\beta = 0. \end{aligned}$$

Up to this point v has been any solution of the associate equation $M(v)=0$. For v we now take the special solution[6]

$$v = (\bar{t} - \alpha)^{(n-1)/2}(\bar{t} - \beta)^{(n-1)/2}, \qquad n \geqq 2.$$

This solution, termed the *resolvent*, is obtained by applying the ordinary method of separation of variables to $M(v)=0$ and plays the role of "Riemann's function." It is convenient to observe that

$\beta = \alpha$ implies $r = 0$, $\alpha = t$,

$$\frac{v_\alpha - v_\beta}{2} = 0, \quad \frac{v_\alpha + v_\beta}{2} = -\frac{n-1}{2}(\bar{t} - t)^{n-2},$$

$\beta = -\alpha$ implies $t = 0$, $\alpha = r$,

$$\frac{v_\alpha - v_\beta}{2} = -\frac{n-1}{2}(\bar{t}^2 - r^2)^{(n-3)/2} r,$$

$$\frac{v_\alpha + v_\beta}{2} = -\frac{n-1}{2}(\bar{t}^2 - r^2)^{(n-3)/2} \bar{t},$$

$\alpha = \bar{t}$ implies $v_\beta = 0$.

More precisely, the last relations hold for $n \geqq 3$, and (8) holds as a result of integrating the fundamental identity (5) over the "wedge" W, all integrals involved being proper integrals. However, if $n=2$ then v_β is infinite on $\alpha = \bar{t}$ and in order to obtain (8)—where improper integrals now appear—it is necessary to integrate first the identity (5) in (α, β, ϕ)-space over the smaller "wedge" $W_{\epsilon,\eta}$ whose cross section in the $\alpha\beta$-plane is bounded by the four straight lines

$$\alpha=\beta, \quad \alpha=-\beta, \quad \beta=\bar{t}-\epsilon, \quad \alpha=\bar{t}-\eta,$$

where $0<\eta<\epsilon<\bar{t}$. Passing to the limit, letting $\eta \to 0$ first, and afterwards letting $\epsilon \to 0$, yields (8).

Thus the last term in (8) drops out altogether, eliminating the need for prescribed data on the characteristic half-cone, and the result is

$$\int_0^{\bar{t}} \int_{\omega_n} (\bar{t} - t)^{n-2} u_t \Big|_{r=0} d\omega_n dt$$
$$= \int_0^{\bar{t}} \int_{\omega_n} \left[(\bar{t}^2 - r^2)^{(n-3)/2} \bar{t} \overset{0}{u_r} + (\bar{t}^2 - r^2)^{(n-3)/2} \cdot r \cdot u^1\right] d\omega_n dr,$$

where the integration on the left is performed on the axis of the cone

[6] Compare G. Darboux, loc. cit., p. 70, for $n=2$.

C. Since

$$\int_0^{\bar{t}} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-2}} dt_{m-1}\left[\int_0^{t_{m-1}} f(t_m)dt_m\right] = \int_0^{\bar{t}} \frac{(\bar{t}-t)^{m-1}}{(m-1)!} f(t)dt,$$

it follows that the preceding relation may be differentiated at least $n-1$ times with respect to $\bar{t}$. Differentiating $n-2$ times with respect to $\bar{t}$ yields the final formula:

$$\begin{aligned} u(P_{\bar{t}}) &= u(P_0) + \frac{1}{(n-2)!\omega_n} \\ &\cdot \frac{\partial^{n-2}}{\partial \bar{t}^{n-2}} \int_0^{\bar{t}} \int_{\omega_n} \left[(\bar{t}^2 - r^2)^{(n-3)/2}\bar{t}u_r^0 + (\bar{t}^2 - r^2)^{(n-3)/2} r u^1\right] d\omega_n dr. \end{aligned} \tag{9}$$

In the present notation, the usual formula[7] for the solution of the Cauchy problem considered above may be written

$$\begin{aligned} u(P_{\bar{t}}) &= \frac{1}{(n-2)!\omega_n} \frac{\partial^{n-1}}{\partial \bar{t}^{n-1}} \int_0^{\bar{t}} \int_{\omega_n} (\bar{t}^2 - r^2)^{(n-3)/2} \cdot r \cdot u^0 d\omega_n dr \\ &+ \frac{1}{(n-2)!\omega_n} \frac{\partial^{n-2}}{\partial \bar{t}^{n-2}} \int_0^{\bar{t}} \int_{\omega_n} (\bar{t}^2 - r^2)^{(n-3)/2} \cdot r \cdot u^1 d\omega_n dr. \end{aligned} \tag{10}$$

The two formulas for $u(P_{\bar{t}})$ are easily seen to coincide,[8] upon differentiating once with respect to $\bar{t}$ the first integral on the right-hand side of (10). This differentiation may be carried out directly under the integral sign if one first sets $r = \bar{t}\rho$. A subsequent integration by parts then yields the result.

In conclusion, the above argument shows the uniqueness of the solution of Cauchy's problem. More precisely, if the Cauchy problem considered has a solution u which possesses continuous second derivatives on $t>0$ and continuous first derivatives on $t \geqq 0$, then u is given by formula (9).

UNIVERSITY OF MARYLAND

[7] R. Courant and D. Hilbert, *Methoden der Mathematische Physik*, vol. II, Berlin, 1937, p. 399.

[8] See M. H. Martin, loc. cit., page 244, for the case $n=2$.