

## THE SECOND HOMOLOGY GROUP OF A GROUP; RELATIONS AMONG COMMUTATORS

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We are concerned with the problem of assigning a group theoretic interpretation to the second homology group  $H_2(G, J)$  of a group  $G$ , with integer coefficients,  $J$  [1, p. 486]. We shall define a new group,  $H(G)$ , called the associated group of  $G$ , which is, roughly speaking, the group of all relations satisfied by commutators in  $G$ , taken modulo those relations which are trivially, or universally, satisfied. (The reader is cautioned not to expect that the associated group of an abelian group necessarily vanishes; we do not regard the statement " $x$  and  $y$  commute implies  $[x, y] = 1$ " as a relation.) We then show that  $H(G) \approx H_2(G, J)$ , so that  $H_2(G, J)$  gives a measure of the extent to which relations among commutators in  $G$  fail to be consequences of universal relations.

For a given group  $G$ , let  $\langle G, G \rangle$  be the free group on all pairs  $\langle x, y \rangle$ , with  $x, y \in G$ . There is a natural homomorphism of  $\langle G, G \rangle$  onto  $[G, G]$  which sends  $\langle x, y \rangle$  into  $[x, y]$ . If  $w \in \langle G, G \rangle$ , we denote its image in  $[G, G]$  by  $[w]$ , and define  $Z(G)$  to be the kernel,

$$Z(G) = \{ w \in \langle G, G \rangle \mid [w] = 1 \}.$$

Let  $B(G)$  be the normal subgroup of  $\langle G, G \rangle$  generated by the relations

- (1)  $\langle x, x \rangle \sim 1,$
- (2)  $\langle x, y \rangle \sim \langle y, x \rangle^{-1},$
- (3)  $\langle xy, z \rangle \sim \langle y, z \rangle^x \langle x, z \rangle,$
- (4)  $\langle y, z \rangle^x \sim \langle x, [y, z] \rangle \langle y, z \rangle,$

where  $x, y,$  and  $z$  range over  $G$  and by definition

$$(5) \quad \langle y, z \rangle^x = \langle y^x, z^x \rangle = \langle xyx^{-1}, xzx^{-1} \rangle.$$

In other words  $B(G)$  is the normal subgroup generated by all  $\langle x, x \rangle$ , all  $\langle x, y \rangle \langle y, x \rangle$ , etc. The symbol  $\sim$  shall mean congruence in  $\langle G, G \rangle$  mod  $B(G)$ . Evidently  $B(G) \subset Z(G)$ , and we define the *associated group* of  $G$  to be

$$H(G) = Z(G)/B(G).$$

If  $h: G \rightarrow G'$  is a homomorphism, we define

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$$h_{\#}: \langle G, G \rangle \rightarrow \langle G', G' \rangle$$

by  $h_{\#}\langle x, y \rangle = \langle h(x), h(y) \rangle$ . Then  $h_{\#}$  carries  $Z(G)$  into  $Z(G')$  and  $B(G)$  into  $B(G')$ , inducing a homomorphism

$$h_*: H(G) \rightarrow H(G'),$$

which satisfies

$$(hg)_* = h_*g_*, \quad 0_* = 0, \quad 1_* = 1,$$

where  $0$  is a zero homomorphism,  $0(x) = 1$ , and  $1$  is an identity homomorphism,  $1(x) = x$ .

By inverting both sides of (3) and quoting (2) we obtain

$$(3') \quad \langle x, yz \rangle \sim \langle x, y \rangle \langle x, z \rangle^y.$$

Of the many consequences of the defining relations of  $B(G)$  we shall have need for only the following:

$$(6) \quad \langle x, y \rangle^{\langle a, b \rangle} \sim \langle x, y \rangle^{\langle a, b \rangle}$$

where  $\langle x, y \rangle^{\langle a, b \rangle}$  is by definition  $\langle a, b \rangle \langle x, y \rangle \langle a, b \rangle^{-1}$ ,

$$(7) \quad [\langle x, y \rangle, \langle a, b \rangle] \sim \langle [x, y], [a, b] \rangle,$$

$$(8) \quad \langle b, b' \rangle \langle a_0, b_0 \rangle \sim \langle [b, b'], a_0 \rangle \langle a_0, [b, b'] b_0 \rangle \langle b, b' \rangle,$$

$$(9) \quad \langle b, b' \rangle \langle b_0, a_0 \rangle \sim \langle [b, b'] b_0, a_0 \rangle \langle a_0, [b, b'] \rangle \langle b, b' \rangle,$$

$$(10) \quad \langle b, b' \rangle \langle a, a' \rangle \sim \langle [b, b'], [a, a'] \rangle \langle a, a' \rangle \langle b, b' \rangle,$$

$$(11) \quad \langle x^n, x^s \rangle \sim 1, \quad n = 0, \pm 1, \dots; s = 0, \pm 1, \dots$$

We prove (6) by expanding  $\langle ax, by \rangle$  in two ways, using (3) and (3'). We have

$$\begin{aligned} \langle ax, by \rangle &\sim \langle ax, b \rangle \langle ax, y \rangle^b \\ &\sim \langle x, b \rangle^a \langle a, b \rangle \langle x, y \rangle^{ba} \langle a, y \rangle^b. \end{aligned}$$

Also

$$\begin{aligned} \langle ax, by \rangle &\sim \langle x, by \rangle^a \langle a, by \rangle \\ &\sim \langle x, b \rangle^a \langle x, y \rangle^{ab} \langle a, b \rangle \langle a, y \rangle^b. \end{aligned}$$

Comparing, we see that

$$\langle a, b \rangle \langle x, y \rangle^{ba} \sim \langle x, y \rangle^{ab} \langle a, b \rangle,$$

or

$$\langle a, b \rangle \langle x, y \rangle^{ba} \langle a, b \rangle^{-1} \sim \langle x, y \rangle^{ab}.$$

Replacing  $x$  and  $y$  by  $x^{(ba)^{-1}}$  and  $y^{(ba)^{-1}}$  gives

$$\langle a, b \rangle \langle x, y \rangle \langle a, b \rangle^{-1} \sim \langle x, y \rangle^{ba(ba)^{-1}} = \langle x, y \rangle^{[a, b]}.$$

Observe that (7) is a consequence of (6), for

$$\begin{aligned} [\langle x, y \rangle, \langle a, b \rangle] &= \langle x, y \rangle^{\langle a, b \rangle} \langle x, y \rangle^{-1} \\ &\sim \langle x, y \rangle^{[a, b]} \langle x, y \rangle^{-1} \\ &\sim \langle [a, b], [x, y] \rangle \langle x, y \rangle \langle x, y \rangle^{-1} \quad \text{by (4).} \end{aligned}$$

Relation (8) is verified by expanding  $\langle a_0, [b, b']b_0 \rangle$  by (3'), giving

$$\begin{aligned} \langle a_0, [b, b']b_0 \rangle &\sim \langle a_0, [b, b'] \rangle \langle a_0, b_0 \rangle^{[b, b']} \\ &\sim \langle a_0, [b, b'] \rangle \langle a_0, b_0 \rangle^{\langle b, b' \rangle} \quad \text{by (6).} \end{aligned}$$

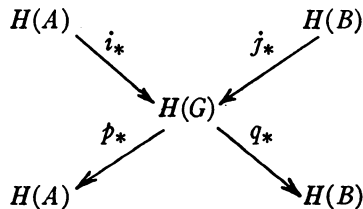
Substitution in the right member of (8) gives the desired result. Relation (9) is proved similarly. Relation (10) is a restatement of (7). Relation (11) is proved, for non-negative  $n$  and  $s$ , by an induction on  $n+s$ , using (3) and (3'). When  $n+s=1$ , say  $n=0$  and  $s=1$ , setting  $x=z$  and  $y=1$  in (3) gives the result. The case of general  $n$  and  $s$  follows trivially from the non-negative case by using (3).

**THEOREM 1.** *The associated group of a free group is a one-element group.*

The case of a free group with an infinite number of generators follows from the case of a free group with a finite number of generators; for if  $F$  is free with infinitely many generators and  $u \in H(F)$ , then  $u \in i_* H(F')$ , where  $F'$  is a subgroup of  $F$  on finitely many generators, and  $i$  is the inclusion. In case the free group  $F$  has but one generator, then  $H(F) = 1$  by virtue of rule (11),  $\langle x^n, x^n \rangle \sim 1$ . The general case of a free group with finitely many generators follows at once by induction from the following

**LEMMA.** *If  $G = A * B$  is the free product of  $A$  and  $B$ , then  $H(G) \approx H(A) \times H(B)$ .*

Let  $i: A \rightarrow G$  and  $j: B \rightarrow G$  be the natural injections, and let  $p: G \rightarrow A$  and  $q: G \rightarrow B$  be the natural projections. Inspection of the diagram



shows that  $i_*$  and  $j_*$  are isomorphisms into and that  $i_* H(A)$  and

$j_*H(B)$  are disjoint. In case  $H(G)$  is the group product  $i_*H(A)j_*H(B)$  the diagram also shows that  $H(G)$  is the direct product  $H(G) = i_*H(A) \times j_*H(B)$ . The problem, then, is to demonstrate that  $H(G) = i_*H(A)j_*H(B)$ .

In order to do this we shall be concerned with three subgroups of  $\langle G, G \rangle: \mathcal{A} = i_{\#} \langle A, A \rangle$ ,  $\mathcal{B} = j_{\#} \langle B, B \rangle$ , and  $\mathcal{M}$ , the subgroup of  $\langle G, G \rangle$  generated by all elements of the form  $\langle a, b \rangle$ , with  $a \neq 1 \in A$ , and  $b \neq 1 \in B$ . Let  $\langle x, y \rangle$  be a generator of  $\langle G, G \rangle$ , with  $x = a_1 b_1 \cdots a_s b_s$ ,  $y = \bar{a}_1 \bar{b}_1 \cdots \bar{a}_r \bar{b}_r$ , and with  $a_i, \bar{a}_j \in A$ ,  $b_i, \bar{b}_j \in B$ . By a repeated application of the product rules (3) and (3') we see that  $\langle x, y \rangle$  is congruent mod  $B(G)$  to a product of elements of the form  $\langle a, a' \rangle^z$ ,  $\langle b, b' \rangle^z$ ,  $\langle a, b \rangle^z$ , and  $\langle b, a \rangle^z$ , with  $a, a' \in A$ ,  $b, b' \in B$ , and  $z \in G$ . Each element of this form can in turn be broken down into a product of terms of the same type, without the exponent  $z$  appearing, by repeated use of the rules

- (5')  $\langle a, a' \rangle^{a_0} = \langle a^{a_0}, a'^{a_0} \rangle$ ,
- (4')  $\langle a, a' \rangle^{b_0} \sim \langle b_0, [a, a'] \rangle \langle a, a' \rangle$ ,
- (12)  $\langle a, b \rangle^{a_0} \sim \langle a_0 a, b \rangle \langle b, a_0 \rangle$ ,
- (13)  $\langle a, b \rangle^{b_0} \sim \langle b_0, a \rangle \langle a, b_0 b \rangle$ ,

and four more similar rules, obtained from these by interchanging  $a$  with  $b$ ,  $a_0$  with  $b_0$ , and  $a'$  with  $b'$ . (5') and (4') are restatements of (5) and (4), and (12) and (13) are restatements of (3) and (3'), using rule (2),  $\langle c, d \rangle^{-1} \sim \langle d, c \rangle$ . Thus we see that  $\langle x, y \rangle$ , and hence any element  $w \in \langle G, G \rangle$ , is congruent to a product  $\pi$  of terms  $\langle a, a' \rangle$ ,  $\langle b, b' \rangle$ ,  $\langle a, b \rangle$ , and  $\langle b, a \rangle$ .

Now take each term  $\langle b, b' \rangle$  in  $\pi$  and "commute" it to the right (beginning with the farthestmost right one and proceeding one at a time) via (8), (9), and (10). Thus we obtain, for the arbitrary element  $w$  of  $\langle G, G \rangle$ ,  $w \sim \pi \sim \pi' \beta$ , with  $\beta$  a product of terms  $\langle b, b' \rangle$ , and  $\pi'$  a product of terms  $\langle a, a' \rangle$ ,  $\langle a, b \rangle$ , and  $\langle b, a \rangle$ . Now take each term of the form  $\langle a, a' \rangle$  in  $\pi'$  and commute it to the left via the rules dual to (8) and (9) (obtained from them by inversion and interchanging  $a$  and  $b$ ); this gives

$$w \sim \pi' \beta \sim \alpha \pi'' \beta,$$

with  $\pi''$  involving only terms  $\langle a, b \rangle$  and  $\langle b, a \rangle$ , and  $\alpha$  a product of terms  $\langle a, a' \rangle$ . By replacing each  $\langle b, a \rangle$  in  $\pi''$  by  $\langle a, b \rangle^{-1}$  we replace  $\pi''$  by  $\mu \in \mathcal{M}$  and have

$$w \sim \alpha \mu \beta$$

with  $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$ , and  $\mu \in \mathcal{M}$ .

Now let  $w \in Z(G)$ , that is,  $[w] = 1$ . Then  $[\alpha][\mu][\beta] = [w] = 1$ , and projecting into  $A$  we see that  $[\alpha] = 1$ ; similarly,  $[\beta] = 1$ , so that  $[\mu] = 1$ . However  $[\mu] = 1$  implies that  $\mu = 1 \in \mathcal{M} \subset \langle G, G \rangle$ . To see this, let  $\mu$  be written as a reduced word in the free group  $\mathcal{M}$ ;  $\mu = \langle a_1, b_1 \rangle^{\epsilon_1} \cdots \langle a_p, b_p \rangle^{\epsilon_p}$ , with  $\epsilon_i = \pm 1, a_i \neq 1 \neq b_i$ . Then, by induction on  $p$ , we see that  $[\mu]$  can be written as a reduced word in the free product  $G = A * B$  in which the last two entries are  $b_p^{-1}a_p^{-1}$  if  $\epsilon_p = -1$ , or  $a_p^{-1}b_p^{-1}$  if  $\epsilon_p = +1$ . In particular  $[\mu] \neq 1$  if  $\mu$  is not the empty word.

Thus  $\mu = 1$  gives  $w \sim \alpha\mu\beta = \alpha\beta$ , with  $[\alpha] = 1$  and  $[\beta] = 1$ , which shows that  $H(G) = i_*H(A)j_*H(B)$  and proves the lemma.

It is possible to use Theorem 1 to show that any “universal relation” among commutators can be deduced from our defining relations (1) to (4). Briefly, a “universal relation” is an expression of the type we have been considering which is valid in any group. We shall not pursue this.

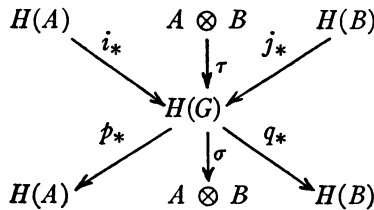
Digressing for a moment, we remark that in proving that  $\langle G, G \rangle$  is the group product, mod  $B(G)$ , of  $\mathcal{A}, \mathcal{M}$ , and  $\mathcal{B}$  we used only the fact that  $G$  is generated by  $A$  and  $B$ . Applying this to the direct product  $G = A \times B$  one sees that  $H(G)$  is the group product of  $i_*H(A), j_*H(B)$ , and  $\tau(A \otimes B)$ , where  $\tau$  is a homomorphism from the tensor product  $A \otimes B$  into  $H(G)$  defined by requiring that  $\tau(a \otimes b)$  is the image in  $H(G)$  of  $\langle a, b \rangle$ . The fact that  $\tau$  is a well defined homomorphism follows from the fact that  $H(G) \approx H_2(G, J)$  is abelian, the congruences

$$\begin{aligned} \langle a_1a_2, b \rangle &\sim \langle a_2, b \rangle^{a_1} \langle a_1, b \rangle && \text{(by (3))} \\ &\sim \langle a_1, 1 \rangle \langle a_2, b \rangle \langle a_1, b \rangle && \text{(by (4))} \\ &\sim \langle a_2, b \rangle \langle a_1, b \rangle && \text{(by (11)),} \end{aligned}$$

and the dual congruence

$$\langle a, b_1b_2 \rangle \sim \langle a, b_1 \rangle \langle a, b_2 \rangle.$$

If we let  $\sigma: H(G) \rightarrow A \otimes B$  be induced by  $\bar{\sigma}: \langle G, G \rangle \rightarrow A \otimes B$ , where  $\bar{\sigma}(\langle a_1b_1, a_2b_2 \rangle) = a_1 \otimes b_2 - a_2 \otimes b_1$ , an analysis of the diagram



shows that  $i_*$ ,  $j_*$ , and  $\tau$  are one-to-one and that

$$\begin{aligned} H(G) &= i_*H(A) \times j_*H(B) \times \tau(A \otimes B) \\ &\approx H(A) \times H(B) \times A \otimes B. \end{aligned}$$

**THEOREM 2.** *There is a canonical isomorphism between  $H(G)$  and  $H_2(G, J)$  preserving the notion of induced homomorphism; if  $h:G \rightarrow G'$  is a homomorphism, we have commutativity in the diagram*

$$\begin{array}{ccc} H(G) & \xrightarrow{h_*} & H(G') \\ \cong & & \cong \\ H_2(G, J) & \xrightarrow{h_*} & H_2(G', J). \end{array}$$

Suppose that  $G$  is given as a factor group of a group  $E$  by a central subgroup  $N$  of  $E$ . The factoring homomorphism  $\eta:E \rightarrow G$  maps  $E$  onto  $G$  with kernel  $N$ . We define a homomorphism  $\langle G, G \rangle \rightarrow E$  by mapping a generator  $\langle x, y \rangle$  of  $\langle G, G \rangle$  into  $[\bar{x}, \bar{y}]$ , where  $\eta(\bar{x})=x$  and  $\eta(\bar{y})=y$ . This is independent of the choices  $\bar{x}$  and  $\bar{y}$  because  $N$  is in the center of  $E$ . This homomorphism carries  $Z(G)$  onto  $N \cap [E, E]$  and carries  $B(G)$  onto 1, and hence induces an onto homomorphism  $\phi:H(G) \rightarrow N \cap [E, E]$ . It is readily verified that the sequence

$$H(E) \xrightarrow{\eta_*} H(G) \xrightarrow{\phi} N \cap [E, E]$$

is exact at  $H(G)$ , that is, kernel  $\phi$  = image  $\eta_*$ .

If  $G$  is an arbitrary group, we can represent  $G$  as the factor group of a free group  $F$  by a subgroup  $R$ ,  $G = F/R$ . Letting  $F^0 = F/[F, R]$  and  $R^0 = R/[F, R]$  we have

$$\begin{array}{ccc} F & \xrightarrow{\lambda} F^0 & \xrightarrow{\eta} G \\ \cup & & \cup \\ R & \longrightarrow & R^0 \\ \cup & & \\ [F, R] & & \end{array}$$

where  $\lambda$  and  $\eta$  are the factoring homomorphisms.  $R^0$  is in the center of  $F^0$ , so that  $\phi$  maps  $H(G)$  onto  $R^0 \cap [F^0, F^0]$ . By exactness in the sequence  $H(F^0) \rightarrow H(G) \rightarrow R^0 \cap [F^0, F^0]$ ,  $\phi$  will be one-to-one provided that  $\eta_* = 0$ . To see that this is actually the case, let  $w = \langle \bar{x}_1, \bar{y}_1 \rangle \cdots \langle \bar{x}_p, \bar{y}_p \rangle \in Z(F^0)$ . Then  $[w] = [x_1, y_1] \cdots [x_p, y_p] = 1 \in F^0$ , and, choosing  $\bar{x}_i, \bar{y}_i \in F$  such that  $\lambda(\bar{x}_i) = x_i$  and  $\lambda(\bar{y}_i) = y_i$ , we have  $\bar{w} = \langle \bar{x}_1, \bar{y}_1 \rangle \cdots \langle \bar{x}_p, \bar{y}_p \rangle$ , with  $\lambda_* \bar{w} = w$ ,  $\lambda[\bar{w}] = [w] = 1$ , and hence  $[\bar{w}] \in [F, R]$ .

Therefore  $[\bar{w}] = [f_1, r_1] \cdots [f_q, r_q]$ , for some  $f_i \in F$  and  $r_i \in R$ . However  $F$  is free,  $H(F) = 1$ , and  $B(F) = Z(F)$ . Hence  $\bar{w} \sim \langle f_1, r_1 \rangle \cdots \langle f_q, r_q \rangle \text{ mod } B(F)$ . Then

$$\begin{aligned} \eta_{\#} w &= \eta_{\#} \lambda_{\#} \bar{w} \sim \eta_{\#} \lambda_{\#} (\langle f_1, r_1 \rangle \cdots \langle f_q, r_q \rangle) \\ &\sim \langle \eta \lambda f_1, 1 \rangle \cdots \langle \eta \lambda f_q, 1 \rangle \sim 1, \end{aligned}$$

and  $\eta_{\#} = 0$ .

Thus  $\phi : H(G) \approx R^0 \cap [F^0, F^0]$ . However  $R^0 \cap [F^0, F^0] = R \cap [F, F] / [F, R]$  is the Hopf construction for  $H_2(G, J)$ , so that we have constructed the desired isomorphism.

A formula can be given for our isomorphism. If  $w = \langle x_1, y_1 \rangle \cdots \langle x_p, y_p \rangle \in Z(G)$ , then the homology class in  $H_2(G, J)$  corresponding to the image of  $w$  in  $H(G)$  is the class of the 2-cycle

$$\begin{aligned} \rho(w) &= \sum_{i=1}^p g(x_i, y_i) \\ (14) \quad &+ \sum_{i=1}^{p-1} \{ ([x_1, y_1] \cdots [x_i, y_i], [x_{i+1}, y_{i+1}]) - (1, 1) \} \end{aligned}$$

where  $g(x, y) = (x, y) - (y, x) - (yx, (yx)^{-1}) + (xy, (yx)^{-1})$ . When  $G$  is abelian this simplifies to

$$(15) \quad \rho(\langle x, y \rangle) = (x, y) - (y, x).$$

A proof of the validity of this formula (we omit it) can be obtained by examining the explicit formulation of the isomorphism  $H_2(G, J) \approx R^0 \cap [F^0, F^0]$  as given by Eilenberg and MacLane [1, p. 485 and 2, p. 75]. (14) shows that the isomorphism is independent of the choice of the representation of  $G$  as  $F/R$  and that commutativity holds in the diagram as asserted in Theorem 2.

As a simple application of our description of  $H_2(A, J)$  we give a more detailed analysis of the structure of  $H(A)$  (and hence of  $H_2(A, J)$  also) for an abelian group  $A$ . Since  $Z(A) = \langle A, A \rangle$ , and  $Z(A)/B(A) = H(A) \approx H_2(A, J)$  is abelian, we see that, for the abelian case only, we may as well have taken  $\langle A, A \rangle$  to be the free abelian group on the pairs  $\langle x, y \rangle$ . This we now do. Writing both  $A$  and  $\langle A, A \rangle$  additively, the defining relations of  $B(A)$ , together with the consequent relation (3'), become

$$\begin{aligned} (1A) \quad &\langle x, x \rangle \sim 0, \\ (2A) \quad &\langle x, y \rangle \sim - \langle y, x \rangle, \\ (3A) \quad &\langle x + y, z \rangle \sim \langle x, z \rangle + \langle y, z \rangle, \end{aligned}$$

$$(3'A) \quad \langle x, y + z \rangle \sim \langle x, y \rangle + \langle x, z \rangle,$$

$$(4A) \quad \langle x, 0 \rangle \sim 0.$$

Observe that (4A) is a consequence of (3'A) by setting  $y=0$  in (3'A). Also,  $\langle x+y, x+y \rangle \sim 0$  by (1A), and expanding this by (3A) and (3'A) shows that (2A) is a consequence of (1A), (3A), and (3'A). Thus  $B(A)$  can be defined by (1A), (3A), and (3'A), which proves:

**THEOREM 3.** *For an abelian group  $A$ ,  $H(A) \approx A \otimes A/D$ , where  $D$  is the subgroup of the tensor product  $A \otimes A$  generated by the diagonal,  $\{a \otimes a \mid a \in A\}$ .*

This gives an isomorphism  $A \otimes A/D \approx H_2(A, J)$ , which, in view of (15), is induced by the homomorphism  $A \otimes A \rightarrow H_2(A, J)$  in which  $x \otimes y$  is mapped into the homology class of  $(x, y) - (y, x)$ .

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