THE SECOND HOMOLOGY GROUP OF A GROUP; RELATIONS AMONG COMMUTATORS

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We are concerned with the problem of assigning a group theoretic interpretation to the second homology group $H_2(G, J)$ of a group G, with integer coefficients, J [1, p. 486]. We shall define a new group, $H(G)$, called the associated group of G, which is, roughly speaking, the group of all relations satisfied by commutators in G, taken modulo those relations which are trivially, or universally, satisfied. (The reader is cautioned not to expect that the associated group of an abelian group necessarily vanishes; we do not regard the statement "x and y commute implies $[x, y]=1$" as a relation.) We then show that $H(G) \approx H_2(G, J)$, so that $H_2(G, J)$ gives a measure of the extent to which relations among commutators in G fail to be consequences of universal relations.

For a given group G, let $\langle G, G\rangle$ be the free group on all pairs $\langle x, y\rangle$, with $x, y \in G$. There is a natural homomorphism of $\langle G, G\rangle$ onto $[G, G]$ which sends $\langle x, y\rangle$ into $[x, y]$. If $w \in \langle G, G\rangle$, we denote its image in $[G, G]$ by $[w]$, and define $Z(G)$ to be the kernel,

$$Z(G) = \{ w \in \langle G, G\rangle \mid [w] = 1 \}.$$

Let $B(G)$ be the normal subgroup of $\langle G, G\rangle$ generated by the relations

$$(1) \qquad \langle x, x\rangle \sim 1,$$

$$(2) \qquad \langle x, y\rangle \sim \langle y, x\rangle^{-1},$$

$$(3) \qquad \langle xy, z\rangle \sim \langle y, z\rangle^{x}\langle x, z\rangle,$$

$$(4) \qquad \langle y, z\rangle^{x} \sim \langle x, [y, z]\rangle\langle y, z\rangle,$$

where x, y, and z range over G and by definition

$$(5) \qquad \langle y, z\rangle^{x} = \langle y^{x}, z^{x}\rangle = \langle xyx^{-1}, xzx^{-1}\rangle.$$

In other words $B(G)$ is the normal subgroup generated by all $\langle x, x\rangle$, all $\langle x, y\rangle\langle y, x\rangle$, etc. The symbol $\sim$ shall mean congruence in $\langle G, G\rangle$ mod $B(G)$. Evidently $B(G) \subset Z(G)$, and we define the *associated group* of G to be

$$H(G) = Z(G)/B(G).$$

If $h: G \to G'$ is a homomorphism, we define

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$$h_\sharp : \langle G, G\rangle \to \langle G', G'\rangle$$

by $h_\sharp\langle x, y\rangle = \langle h(x), h(y)\rangle$. Then $h_\sharp$ carries $Z(G)$ into $Z(G')$ and $B(G)$ into $B(G')$, inducing a homomorphism

$$h_* : H(G) \to H(G'),$$

which satisfies

$$(hg)_* = h_* g_*, \qquad 0_* = 0, \qquad 1_* = 1,$$

where 0 is a zero homomorphism, $0(x) = 1$, and 1 is an identity homomorphism, $1(x) = x$.

By inverting both sides of (3) and quoting (2) we obtain

(3′) $$\langle x, yz\rangle \sim \langle x, y\rangle\langle x, z\rangle^{y}.$$

Of the many consequences of the defining relations of $B(G)$ we shall have need for only the following:

(6) $$\langle x, y\rangle^{\langle a,b\rangle} \sim \langle x, y\rangle^{[a,b]}$$

where $\langle x, y\rangle^{\langle a,b\rangle}$ is by definition $\langle a, b\rangle\langle x, y\rangle\langle a, b\rangle^{-1}$,

(7) $[\langle x, y\rangle, \langle a, b\rangle] \sim \langle [x, y], [a, b]\rangle$,

(8) $\langle b, b'\rangle\langle a_0, b_0\rangle \sim \langle [b, b'], a_0\rangle\langle a_0, [b, b']b_0\rangle\langle b, b'\rangle$,

(9) $\langle b, b'\rangle\langle b_0, a_0\rangle \sim \langle [b, b']b_0, a_0\rangle\langle a_0, [b, b']\rangle\langle b, b'\rangle$,

(10) $\langle b, b'\rangle\langle a, a'\rangle \sim \langle [b, b'], [a, a']\rangle\langle a, a'\rangle\langle b, b'\rangle$,

(11) $\langle x^n, x^s\rangle \sim 1, \qquad n = 0, \pm 1, \cdots; s = 0, \pm 1, \cdots.$

We prove (6) by expanding $\langle ax, by\rangle$ in two ways, using (3) and (3′). We have

$$\begin{aligned}\langle ax, by\rangle &\sim \langle ax, b\rangle\langle ax, y\rangle^{b}\\ &\sim \langle x, b\rangle^{a}\langle a, b\rangle\langle x, y\rangle^{ba}\langle a, y\rangle^{b}.\end{aligned}$$

Also

$$\begin{aligned}\langle ax, by\rangle &\sim \langle x, by\rangle^{a}\langle a, by\rangle\\ &\sim \langle x, b\rangle^{a}\langle x, y\rangle^{ab}\langle a, b\rangle\langle a, y\rangle^{b}.\end{aligned}$$

Comparing, we see that

$$\langle a, b\rangle\langle x, y\rangle^{ba} \sim \langle x, y\rangle^{ab}\langle a, b\rangle,$$

or

$$\langle a, b\rangle\langle x, y\rangle^{ba}\langle a, b\rangle^{-1} \sim \langle x, y\rangle^{ab}.$$

Replacing x and y by $x^{(ba)^{-1}}$ and $y^{(ba)^{-1}}$ gives

$$\langle a, b\rangle\langle x, y\rangle\langle a, b\rangle^{-1} \sim \langle x, y\rangle^{ba(ba)^{-1}} = \langle x, y\rangle^{[a,b]}.$$

Observe that (7) is a consequence of (6), for

$$\begin{aligned}[\langle x, y\rangle, \langle a, b\rangle] &= \langle x, y\rangle^{\langle a,b\rangle}\langle x, y\rangle^{-1} \\ &\sim \langle x, y\rangle^{[a,b]}\langle x, y\rangle^{-1} \\ &\sim \langle [a, b], [x, y]\rangle\langle x, y\rangle\langle x, y\rangle^{-1} \qquad \text{by (4).}\end{aligned}$$

Relation (8) is verified by expanding $\langle a_0, [b, b']b_0\rangle$ by (3′), giving

$$\begin{aligned}\langle a_0, [b, b']b_0\rangle &\sim \langle a_0, [b, b']\rangle\langle a_0, b_0\rangle^{[b,b']} \\ &\sim \langle a_0, [b, b']\rangle\langle a_0, b_0\rangle^{\langle b,b'\rangle} \qquad \text{by (6).}\end{aligned}$$

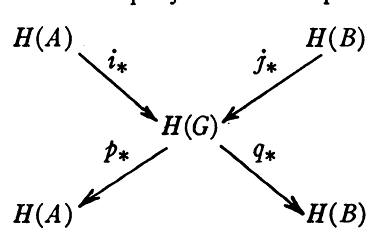
Substitution in the right member of (8) gives the desired result. Relation (9) is proved similarly. Relation (10) is a restatement of (7). Relation (11) is proved, for non-negative n and s, by an induction on $n+s$, using (3) and (3′). When $n+s=1$, say $n=0$ and $s=1$, setting $x=z$ and $y=1$ in (3) gives the result. The case of general n and s follows trivially from the non-negative case by using (3).

THEOREM 1. *The associated group of a free group is a one-element group.*

The case of a free group with an infinite number of generators follows from the case of a free group with a finite number of generators; for if F is free with infinitely many generators and $u \in H(F)$, then $u \in i_*H(F')$, where F' is a subgroup of F on finitely many generators, and i is the inclusion. In case the free group F has but one generator, then $H(F)=1$ by virtue of rule (11), $\langle x^n, x^s\rangle \sim 1$. The general case of a free group with finitely many generators follows at once by induction from the following

LEMMA. *If $G=A * B$ is the free product of A and B, then $H(G) \approx H(A) \times H(B)$.*

Let $i:A \to G$ and $j:B \to G$ be the natural injections, and let $p:G \to A$ and $q:G \to B$ be the natural projections. Inspection of the diagram



shows that i_* and j_* are isomorphisms into and that $i_*H(A)$ and

$j_*H(B)$ are disjoint. In case $H(G)$ is the group product $i_*H(A)j_*H(B)$ the diagram also shows that $H(G)$ is the direct product $H(G) = i_*H(A)\times j_*H(B)$. The problem, then, is to demonstrate that $H(G) = i_*H(A)j_*H(B)$.

In order to do this we shall be concerned with three subgroups of $\langle G, G\rangle$: $\mathcal{A}=i_\#\langle A, A\rangle$, $\mathcal{B}=j_\#\langle B, B\rangle$, and $\mathcal{M}$, the subgroup of $\langle G, G\rangle$ generated by all elements of the form $\langle a, b\rangle$, with $a\neq 1\in A$, and $b\neq 1\in B$. Let $\langle x, y\rangle$ be a generator of $\langle G, G\rangle$, with $x=a_1b_1\cdots a_sb_s$, $y=\bar{a}_1\bar{b}_1\cdots\bar{a}_r\bar{b}_r$, and with $a_i, \bar{a}_j\in A$, $b_i, \bar{b}_j\in B$. By a repeated application of the product rules (3) and (3′) we see that $\langle x, y\rangle$ is congruent mod $B(G)$ to a product of elements of the form $\langle a, a'\rangle^z$, $\langle b, b'\rangle^z$, $\langle a, b\rangle^z$, and $\langle b, a\rangle^z$, with $a, a'\in A$, $b, b'\in B$, and $z\in G$. Each element of this form can in turn be broken down into a product of terms of the same type, without the exponent z appearing, by repeated use of the rules

$$\langle a, a'\rangle^{a_0} = \langle a^{a_0}, a'^{a_0}\rangle, \tag{5′}$$

$$\langle a, a'\rangle^{b_0} \sim \langle b_0, [a, a']\rangle\langle a, a'\rangle, \tag{4′}$$

$$\langle a, b\rangle^{a_0} \sim \langle a_0a, b\rangle\langle b, a_0\rangle, \tag{12}$$

$$\langle a, b\rangle^{b_0} \sim \langle b_0, a\rangle\langle a, b_0b\rangle, \tag{13}$$

and four more similar rules, obtained from these by interchanging a with b, a_0 with b_0, and a' with b'. (5′) and (4′) are restatements of (5) and (4), and (12) and (13) are restatements of (3) and (3′), using rule (2), $\langle c, d\rangle^{-1}\sim\langle d, c\rangle$. Thus we see that $\langle x, y\rangle$, and hence any element $w\in\langle G, G\rangle$, is congruent to a product π of terms $\langle a, a'\rangle$, $\langle b, b'\rangle$, $\langle a, b\rangle$, and $\langle b, a\rangle$.

Now take each term $\langle b, b'\rangle$ in π and "commute" it to the right (beginning with the farthermost right one and proceeding one at a time) via (8), (9), and (10). Thus we obtain, for the arbitrary element w of $\langle G, G\rangle$, $w\sim\pi\sim\pi'\beta$, with β a product of terms $\langle b, b'\rangle$, and π' a product of terms $\langle a, a'\rangle$, $\langle a, b\rangle$, and $\langle b, a\rangle$. Now take each term of the form $\langle a, a'\rangle$ in π' and commute it to the left via the rules dual to (8) and (9) (obtained from them by inversion and interchanging a and b); this gives

$$w \sim \pi'\beta \sim \alpha\pi''\beta,$$

with π'' involving only terms $\langle a, b\rangle$ and $\langle b, a\rangle$, and α a product of terms $\langle a, a'\rangle$. By replacing each $\langle b, a\rangle$ in π'' by $\langle a, b\rangle^{-1}$ we replace π'' by $\mu\in\mathcal{M}$ and have

$$w \sim \alpha\mu\beta$$

with $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$, and $\mu \in \mathcal{M}$.

Now let $w \in Z(G)$, that is, $[w]=1$. Then $[\alpha][\mu][\beta]=[w]=1$, and projecting into A we see that $[\alpha]=1$; similarly, $[\beta]=1$, so that $[\mu]=1$. However $[\mu]=1$ implies that $\mu=1\in\mathcal{M}\subset\langle G, G\rangle$. To see this, let μ be written as a reduced word in the free group $\mathcal{M}$; $\mu=\langle a_1, b_1\rangle^{\epsilon_1}\cdots\langle a_p, b_p\rangle^{\epsilon_p}$, with $\epsilon_i=\pm 1$, $a_i\neq 1\neq b_i$. Then, by induction on p, we see that $[\mu]$ can be written as a reduced word in the free product $G=A*B$ in which the last two entries are $b_p^{-1}a_p^{-1}$ if $\epsilon_p=-1$, or $a_p^{-1}b_p^{-1}$ if $\epsilon_p=+1$. In particular $[\mu]\neq 1$ if μ is not the empty word.

Thus $\mu=1$ gives $w\sim\alpha\mu\beta=\alpha\beta$, with $[\alpha]=1$ and $[\beta]=1$, which shows that $H(G)=i_*H(A)j_*H(B)$ and proves the lemma.

It is possible to use Theorem 1 to show that any "universal relation" among commutators can be deduced from our defining relations (1) to (4). Briefly, a "universal relation" is an expression of the type we have been considering which is valid in any group. We shall not pursue this.

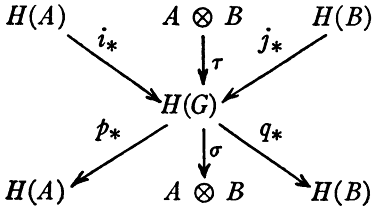
Digressing for a moment, we remark that in proving that $\langle G, G\rangle$ is the group product, mod $B(G)$, of $\mathcal{A}$, $\mathcal{M}$, and $\mathcal{B}$ we used only the fact that G is generated by A and B. Applying this to the direct product $G=A\times B$ one sees that $H(G)$ is the group product of $i_*H(A)$, $j_*H(B)$, and $\tau(A\otimes B)$, where τ is a homomorphism from the tensor product $A\otimes B$ into $H(G)$ defined by requiring that $\tau(a\otimes b)$ is the image in $H(G)$ of $\langle a, b\rangle$. The fact that τ is a well defined homomorphism follows from the fact that $H(G)\approx H_2(G, J)$ is abelian, the congruences

$$\begin{aligned}\langle a_1a_2, b\rangle &\sim \langle a_2, b\rangle^{a_1}\langle a_1, b\rangle && \text{(by (3))}\\ &\sim \langle a_1, 1\rangle\langle a_2, b\rangle\langle a_1, b\rangle && \text{(by (4))}\\ &\sim \langle a_2, b\rangle\langle a_1, b\rangle && \text{(by (11))},\end{aligned}$$

and the dual congruence

$$\langle a, b_1b_2\rangle \sim \langle a, b_1\rangle\langle a, b_2\rangle.$$

If we let $\sigma: H(G)\to A\otimes B$ be induced by $\bar\sigma:\langle G, G\rangle\to A\otimes B$, where $\bar\sigma(\langle a_1b_1, a_2b_2\rangle)=a_1\otimes b_2-a_2\otimes b_1$, an analysis of the diagram



shows that i_*, j_*, and τ are one-to-one and that

$$H(G) = i_*H(A) \times j_*H(B) \times \tau(A \otimes B)$$
$$\approx H(A) \times H(B) \times A \otimes B.$$

THEOREM 2. *There is a canonical isomorphism between $H(G)$ and $H_2(G, J)$ preserving the notion of induced homomorphism; if $h:G\to G'$ is a homomorphism, we have commutativity in the diagram*

$$\begin{array}{ccc} H(G) & \xrightarrow{h_*} & H(G') \\ \wr & & \wr \\ H_2(G, J) & \xrightarrow{h_*} & H_2(G', J). \end{array}$$

Suppose that G is given as a factor group of a group E by a central subgroup N of E. The factoring homomorphism $\eta:E\to G$ maps E onto G with kernel N. We define a homomorphism $\langle G, G\rangle\to E$ by mapping a generator $\langle x, y\rangle$ of $\langle G, G\rangle$ into $[\bar{x}, \bar{y}]$, where $\eta(\bar{x})=x$ and $\eta(\bar{y})=y$. This is independent of the choices $\bar{x}$ and $\bar{y}$ because N is in the center of E. This homomorphism carries $Z(G)$ onto $N\cap[E, E]$ and carries $B(G)$ onto 1, and hence induces an onto homomorphism $\phi:H(G)\to N\cap[E, E]$. It is readily verified that the sequence

$$H(E) \xrightarrow{\eta_*} H(G) \xrightarrow{\phi} N \cap [E, E]$$

is exact at $H(G)$, that is, kernel ϕ = image η_*.

If G is an arbitrary group, we can represent G as the factor group of a free group F by a subgroup R, $G=F/R$. Letting $F^0=F/[F, R]$ and $R^0=R/[F, R]$ we have

$$\begin{array}{ccccc} F & \xrightarrow{\lambda} & F^0 & \xrightarrow{\eta} & G \\ \cup & & \cup & & \\ R & \longrightarrow & R^0 & & \\ \cup & & & & \\ [F, R] & & & & \end{array}$$

where λ and η are the factoring homomorphisms. R^0 is in the center of F^0, so that ϕ maps $H(G)$ onto $R^0\cap[F^0, F^0]$. By exactness in the sequence $H(F^0)\to H(G)\to R^0\cap[F^0, F^0]$, ϕ will be one-to-one provided that $\eta_*=0$. To see that this is actually the case, let $w=\langle \bar{x}_1, y_1\rangle \cdots \langle x_p, y_p\rangle\in Z(F^0)$. Then $[w]=[x_1, y_1]\cdots[x_p, y_p]=1\in F^0$, and, choosing $\bar{x}_i, \bar{y}_i\in F$ such that $\lambda(\bar{x}_i)=x_i$ and $\lambda(\bar{y}_i)=y_i$, we have $\bar{w}=\langle x_1,\bar{y}_1\rangle\cdots\langle\bar{x}_p, \bar{y}_p\rangle$, with $\lambda_\sharp\bar{w}=w$, $\lambda[\bar{w}]=[w]=1$, and hence $[\bar{w}]\in[F, R]$.

Therefore $[\bar{w}]=[f_1, r_1] \cdots [f_q, r_q]$, for some $f_i \in F$ and $r_i \in R$. However F is free, $H(F)=1$, and $B(F)=Z(F)$. Hence $\bar{w} \sim \langle f_1, r_1\rangle \cdots \langle f_q, r_q\rangle$ mod $B(F)$. Then

$$\begin{aligned} \eta_\# w = \eta_\# \lambda_\# \bar{w} &\sim \eta_\# \lambda_\# (\langle f_1, r_1\rangle \cdots \langle f_q, r_q\rangle) \\ &\sim \langle \eta\lambda f_1, 1\rangle \cdots \langle \eta\lambda f_q, 1\rangle \sim 1, \end{aligned}$$

and $\eta_* = 0$.

Thus $\phi: H(G) \approx R^0 \cap [F^0, F^0]$. However $R^0 \cap [F^0, F^0] = R \cap [F, F]/[F, R]$ is the Hopf construction for $H_2(G, J)$, so that we have constructed the desired isomorphism.

A formula can be given for our isomorphism. If $w = \langle x_1, y_1\rangle \cdots \langle x_p, y_p\rangle \in Z(G)$, then the homology class in $H_2(G, J)$ corresponding to the image of w in $H(G)$ is the class of the 2-cycle

$$\begin{aligned} \rho(w) = &\sum_{i=1}^{p} g(x_i, y_i) \\ &+ \sum_{i=1}^{p-1} \{([x_1, y_1] \cdots [x_i, y_i], [x_{i+1}, y_{i+1}]) - (1, 1)\} \end{aligned} \tag{14}$$

where $g(x, y) = (x, y) - (y, x) - (yx, (yx)^{-1}) + (xy, (yx)^{-1})$. When G is abelian this simplifies to

$$\rho(\langle x, y\rangle) = (x, y) - (y, x). \tag{15}$$

A proof of the validity of this formula (we omit it) can be obtained by examining the explicit formulation of the isomorphism $H_2(G, J) \approx R^0 \cap [F^0, F^0]$ as given by Eilenberg and MacLane [**1**, p. 485 and **2**, p. 75]. (14) shows that the isomorphism is independent of the choice of the representation of G as F/R and that commutativity holds in the diagram as asserted in Theorem 2.

As a simple application of our description of $H_2(A, J)$ we give a more detailed analysis of the structure of $H(A)$ (and hence of $H_2(A, J)$ also) for an abelian group A. Since $Z(A) = \langle A, A\rangle$, and $Z(A)/B(A) = H(A) \approx H_2(A, J)$ is abelian, we see that, for the abelian case only, we may as well have taken $\langle A, A\rangle$ to be the free abelian group on the pairs $\langle x, y\rangle$. This we now do. Writing both A and $\langle A, A\rangle$ additively, the defining relations of $B(A)$, together with the consequent relation (3′), become

$$\langle x, x\rangle \sim 0, \tag{1A}$$

$$\langle x, y\rangle \sim -\langle y, x\rangle, \tag{2A}$$

$$\langle x + y, z\rangle \sim \langle x, z\rangle + \langle y, z\rangle, \tag{3A}$$

$$\langle x, y + z \rangle \sim \langle x, y \rangle + \langle x, z \rangle, \tag{3'A}$$

$$\langle x, 0 \rangle \sim 0. \tag{4A}$$

Observe that (4A) is a consequence of (3'A) by setting $y=0$ in (3'A). Also, $\langle x+y, x+y\rangle \sim 0$ by (1A), and expanding this by (3A) and (3'A) shows that (2A) is a consequence of (1A), (3A), and (3'A). Thus $B(A)$ can be defined by (1A), (3A), and (3'A), which proves:

THEOREM 3. *For an abelian group A, $H(A) \approx A \otimes A/D$, where D is the subgroup of the tensor product $A \otimes A$ generated by the diagonal, $\{a \otimes a \mid a \in A\}$.*

This gives an isomorphism $A \otimes A/D \approx H_2(A, J)$, which, in view of (15), is induced by the homomorphism $A \otimes A \rightarrow H_2(A, J)$ in which $x \otimes y$ is mapped into the homology class of $(x, y) - (y, x)$.

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