

QUADRATIC DIAMETER OF A METRIC SPACE AND ITS APPLICATION TO A PROBLEM IN ANALYSIS

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1. **Introduction.** The purpose of this note is to introduce the notion of quadratic diameter and to apply it to a problem in analysis concerning the measure of the set of values taken by a real-valued differentiable function on its critical set. It was proved by A. P. Morse [3] that if $f(x)$ is a real-valued function defined on an open set of an n -dimensional Euclidean space which is n -times continuously partially differentiable, then the set of values taken by $f(x)$ on its critical set (i.e. the set of all x at which all first partial derivatives of $f(x)$ vanish) is a set of one-dimensional Lebesgue measure zero. We shall give a simple proof of this result of Morse for the case $n = 2$, under a weaker assumption that the first partial derivatives of $f(x)$ satisfy the Lipschitz condition. (It is mentioned in [3] that his condition can be weakened and that it is sufficient to assume that all $(n - 1)$ th partial derivatives of $f(x)$ are totally differentiable; but this weakened condition is still stronger than our condition for the case $n = 2$.) It is, however, to be observed that our method seems to work only for the case $n = 2$, and the problem of discussing the possibility of similar approach for the higher-dimensional cases $n > 2$ is still open.

2. **Quadratic diameter.** Let $R = \{x, y, \dots\}$ be a metric space with the metric $d(x, y)$. For any subset S of R and for any two points x, y of S , the *quadratic distance* $d^{(2)}(x, y; S)$ of x, y with respect to S is defined by

$$(2.1) \quad d^{(2)}(x, y, S) = \inf \sum_{i=1}^p (d(x_{i-1}, x_i))^2,$$

where inf is taken for all possible finite *chains* $x_0 = x, x_1, \dots, x_p = y$ from S . (The *length* p of this chain is also arbitrary.) Further, the *quadratic diameter* $\delta^{(2)}(S)$ of S is defined by

$$(2.2) \quad \delta^{(2)}(S) = \sup_{x, y \in S} d^{(2)}(x, y; S).$$

Finally, the *quadratic variation* $v^{(2)}(S)$ of S is defined by

$$(2.3) \quad v^{(2)}(S) = \inf \sum_{i=1}^p \delta^{(2)}(S_i),$$

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where inf is taken for all possible decompositions $S = \cup_{i=1}^p S_i$ of S into a finite number of (not necessarily disjoint) closed subsets S_i , $i=1, \dots, p$.

We begin with simple observations concerning the quantities defined above.

LEMMA 2.1.

$$(2.4) \quad d^{(2)}(x, y; S) = d^{(2)}(y, x; S) \leq (d(x, y))^2,$$

$$(2.5) \quad d^{(2)}(x, z; S) \leq d^{(2)}(x, y; S) + d^{(2)}(y, z; S)$$

for any $x, y, z \in S$.

LEMMA 2.2.

$$(2.6) \quad v^{(2)}(S) \leq \delta^{(2)}(S) \leq (\delta(S))^2,$$

where $\delta(S) = \sup_{x, y \in S} d(x, y)$ is the ordinary diameter of S . Further

$$(2.7) \quad \delta^{(2)}(S_1 \cup S_2) \leq \delta^{(2)}(S_1) + \delta^{(2)}(S_2)$$

if S_1 and S_2 are not disjoint, and

$$(2.8) \quad v^{(2)}(S) = \delta^{(2)}(S)$$

if S is connected.

The proofs of these lemmas are easy and so omitted.

Because of Lemma 2.1, $d^{(2)}(x, y; S)$ may be considered as a quasi-metric on S . It is possible that $d^{(2)}(x, y; S) = 0$ for two different points x, y of S (as will be shown in Lemma 2.3). In this case x, y are called *$d^{(2)}$ -equivalent*. It is easy to see that this relation of $d^{(2)}$ -equivalence is symmetric and transitive, and that for any $x \in S$ the set $E(x)$ of all $y \in S$ which is $d^{(2)}$ -equivalent with x is a closed subset of S . Further, $d^{(2)}(E(x), E(y)) = d^{(2)}(x, y; S)$ may be considered as a metric defined on the class of all equivalence classes $E(x)$ of S .

It is to be noticed that $S_1 \subset S_2 \subset R$ and $x, y \in S_1$ imply

$$(2.9) \quad d^{(2)}(x, y; S_1) \geq d^{(2)}(x, y; S_2),$$

and that it is possible that the inequality holds in (2.9). Thus the quadratic diameter $\delta^{(2)}(S)$ of S is not necessarily a monotone set function of S .

We do not need the following Lemmas 2.3, 2.4 in the proof of our main result in §4. But a similar argument plays an important role in the proof of Lemma 4.3.

LEMMA 2.3. *If S is a subset of an n -dimensional Euclidean space R^n ($n \geq 1$), and if x, y are two points of S such that the segment $[x, y]$*

connecting x and y is contained in S , then $d^{(2)}(x, y; S) = 0$.

PROOF. For any integer $p \geq 1$, let $x_0 = x, x_1, \dots, x_p = y$ be a chain of points from S which divide the segment $[x, y]$ into p equal parts. Then $\sum_{i=1}^p (d(x_{i-1}, x_i))^2 = (1/p)(d(x, y))^2$, and this implies that $d^{(2)}(x, y; S) = 0$.

COROLLARY. If S is a convex subset of a Euclidean space R^n ($n \geq 1$), then $\delta^{(2)}(S) = 0$.

LEMMA 2.4. If x, y are two points of a subset S of a metric space R which can be connected by a rectifiable curve in S , then $d^{(2)}(x, y; S) = 0$.

PROOF. Let $C = \{x_t | 0 \leq t \leq 1\}$ ($x_0 = x, x_1 = y$) be a rectifiable curve which connects x and y in S . Let l be the curve length of C . Then, for any $\epsilon > 0$, there exist real numbers t_0, t_1, \dots, t_p such that $t_0 = 0 < t_1 < \dots < t_p = 1, d(x_{i-1}, x_i) < \epsilon, i = 1, \dots, p$, and

$$(2.10) \quad \sum_{i=1}^p d(x_{t_{i-1}}, x_{t_i}) < l + \epsilon.$$

From this follows

$$(2.11) \quad \sum_{i=1}^p (d(x_{t_{i-1}}, x_{t_i}))^2 < \epsilon(l + \epsilon)$$

and this implies $d^{(2)}(x, y; S) = 0$.

COROLLARY. If S is a subset of a metric space R such that any two points x, y of S can be connected by a rectifiable curve in S , then $\delta^{(2)}(S) = 0$.

The question then arises to determine, given a metric space R , whether or not there exists a connected subset S of R with the property that $d^{(2)}(x, y; S) > 0$ for any two different points x, y of S . We shall show in §3 that in a Hilbert space or in a Euclidean space R^n with $n \geq 3$ there exists a simple curve $S = \{x_t | 0 \leq t \leq 1\}$ such that $d^{(2)}(x_s, x_t; S) > 0$ for any s, t with $0 \leq s < t \leq 1$.

3. Examples of continuous curves with a positive quadratic diameter. Let $S = \{x_t | 0 \leq t \leq 1\}$ be a curve in a Hilbert space R with the following properties: (i) the vectors $x_t - x_s$ and $x_{t'} - x_{s'}$ are orthogonal if $0 \leq s < t \leq s' < t' \leq 1$, (ii) $d(x_s, x_t) = (t-s)^{1/2}$ if $0 \leq s < t \leq 1$. Then it is easy to see that

$$(3.1) \quad \sum_{i=1}^p (d(x_{t_{i-1}}, x_{t_i}))^2 = (d(x_s, x_t))^2 = t - s$$

for any $s, t_0, t_1, \dots, t_p, t$ such that $0 \leq s = t_0 < t_1 < \dots < t_p = t \leq 1$. From this follows that

$$(3.2) \quad d^{(2)}(x_s, x_t; S) = t - s$$

for any s, t with $0 \leq s < t \leq 1$. An example of a curve S in a Hilbert space with the properties (i), (ii) can be given as follows: Let $R = L^2(0, 1)$ be the L^2 -space of all real-valued square integrable functions $x(u)$ defined on the unit interval $I = \{u | 0 \leq u \leq 1\}$ with the usual norm

$$(3.3) \quad \|x\| = \left(\int_0^1 |x(u)|^2 du \right)^{1/2} < \infty,$$

and let $S = \{x_t | 0 \leq t \leq 1\}$ be a curve in R defined by

$$(3.4) \quad x_t(u) = \begin{cases} 1, & 0 \leq u \leq t, \\ 0, & t < u \leq 1. \end{cases}$$

Then it is easy to see that S has the properties (i), (ii) above. It is to be noticed that from Lemma 2.4 follows that S is not rectifiable. In fact it is easy to see that

$$\sum_{i=1}^p d(x_{(i-1)/p}, x_{i/p}) = p \cdot \frac{1}{p^{1/2}} = p^{1/2} \rightarrow \infty$$

as $p \rightarrow \infty$. Such curves have been discussed by N. Wiener [5], A. Kolmogoroff [1], and P. Lévy [2] in connection with the theory of Brownian motion.

In case R is a finite-dimensional Euclidean space R^n ($n \geq 3$), the existence of a curve $S = \{x_t | 0 \leq t \leq 1\}$ with the same property can be shown by using a Peano curve. In fact, let $\{(u_i, v_i) | 0 \leq t \leq 1\}$ be the usual parametrization of the standard Peano curve filling the closed unit square $\bar{Q} = \{(u, v) | 0 \leq u, v \leq 1\}$ of R^2 with the property that, for any $n = 1, 2, \dots$ and for any $i = 1, \dots, 2^{2n}$, the image of the closed interval

$$(3.5) \quad \bar{I}_i^{(n)} = \{t | (i-1)2^{-2n} \leq t \leq i \cdot 2^{-2n}\}$$

is one of the 2^{2n} closed squares

$$(3.6) \quad \bar{Q}_{k,l}^{(n)} = \{(u, v) | (k-1)2^{-n} \leq u \leq k \cdot 2^{-n}, (l-1)2^{-n} \leq v \leq l \cdot 2^{-n}\}, \\ k, l = 1, \dots, 2^n.$$

Then $S = \{x_t = (u_t, v_t, t) | 0 \leq t \leq 1\}$ is a simple curve in R^3 whose

projection on the (u, v) -plane fills up the closed unit interval $\bar{Q} = \{(u, v) \mid 0 \leq u, v \leq 1\}$. It is easy to see that $d^{(2)}(x_s, x_t; S) > 0$ for any s, t with $0 \leq s < t \leq 1$.

On the other hand, it is impossible to construct similar examples in R^2 . In fact, as will be shown in Theorem 2 below, $\delta^{(2)}(S) = 0$ for any compact connected subset S of R^2 and hence $d^{(2)}(x, y; S) = 0$ for any two points x, y of a continuous curve S in $R^{(2)}$.

If we consider the case when S is compact but not connected, then it is clear that $\delta^{(2)}(S) > 0$. In fact, if $S = S_1 \cup S_2$ is a decomposition of S into two disjoint compact subsets S_1 and S_2 , then there exists a constant $\eta > 0$ such that $d(x, y) \geq \eta > 0$ for any $x \in S_1, y \in S_2$ and hence $\delta^{(2)}(S) \geq d^{(2)}(x, y; S) \geq \eta^2 > 0$ for any $x \in S_1, y \in S_2$. Thus it seems as if there is nothing we can conclude about $\delta^{(2)}(S)$ in general. But we can show that if S is a compact subset of R^2 , then the quadratic variation $v^2(S)$ of S is zero, or in other words, that, for any $\eta > 0$, there exists a decomposition $S = \bigcup_{i=1}^p S_i$ of S into a finite number of subsets $S_i, i = 1, \dots, p$, such that $\sum_{i=1}^p \delta^{(2)}(S_i) < \eta$. This will be proved in Theorem 1.

4. Quadratic diameter of a set in R^2 .

LEMMA 4.1. *Let S be a compact subset of a two-dimensional Euclidean space R^2 . If we denote by αS a subset of R^2 which is homothetic with S with the ratio $\alpha:1$, then*

$$(4.1) \quad \delta^{(2)}(\alpha S) = \alpha^2 \delta^{(2)}(S).$$

PROOF. The proof is clear from definition.

LEMMA 4.2. *Let S be a compact subset of the closed unit square $\bar{Q} = \{(u, v) \mid 0 \leq u, v \leq 1\}$ of R^2 with $m^{(2)}(S) < \epsilon$, where $m^{(2)}(S)$ denotes the two-dimensional Lebesgue measure of S . Then there exists a decomposition $S = \bigcup_{i=1}^p S_i$ of S into a finite number of closed subsets $S_i, i = 1, \dots, p$, such that*

$$(4.2) \quad \sum_{i=1}^p \delta^{(2)}(S_i) < 2\epsilon.$$

PROOF. Since $m^{(2)}(S) < \epsilon$, there exists an open set O of R^2 such that $S \subset O$ and $m^{(2)}(O) < \epsilon$. Since O is open, there exists a countable number of open squares $Q_i, i = 1, 2, \dots$, such that $O = \bigcup_{i=1}^{\infty} Q_i$ and $\sum_{i=1}^{\infty} m^{(2)}(Q_i) < \epsilon$. Since S is compact, there exists a positive integer p such that $S \subset \bigcup_{i=1}^p Q_i$. If we put $S_i = S \cap \bar{Q}_i$ where \bar{Q}_i is the closure of $Q_i, i = 1, \dots, p$, then $S = \bigcup_{i=1}^p S_i$ is a required decomposition. In fact, in the notation of (2.6), it is clear that $\sum_{i=1}^p \delta^{(2)}(S_i)$

$$\leq \sum_{i=1}^p (\delta(S_i))^2 \leq \sum_{i=1}^p (\delta(\bar{Q}_i))^2 = 2 \sum_{i=1}^p m^{(2)}(Q_i) < 2 \sum_{i=1}^{\infty} m^{(2)}(Q_i) < 2\epsilon.$$

LEMMA 4.3. Let S be a compact subset of the closed unit square $\bar{Q} = \{(u, v) \mid 0 \leq u, v \leq 1\}$ of R^2 such that $m^{(2)}(\bar{Q} - S) < \epsilon$, where $\epsilon < \pi/16$ and $m^{(2)}(\bar{Q} - S)$ denotes the two-dimensional Lebesgue measure of $\bar{Q} - S$. Then

$$(4.3) \quad \delta^{(2)}(S) < 50 \epsilon^{1/2}.$$

PROOF. Let p be a positive integer such that $p^{-1} \leq \epsilon^{1/2} < (p-1)^{-1}$. For any two points x, y of S , let $x_0 = x, x_1, \dots, x_p = y$ be the chain of points from \bar{Q} of length p which divides the segment $[x, y]$ in p equal parts. We do not know whether these dividing points x_1, \dots, x_{p-1} belong to S or not. But it is possible to show that for each $i = 1, \dots, p-1$, there exists a point x'_i from S such that $d(x_i, x'_i) < 2(\epsilon/\pi)^{1/2}$. In fact, if we denote by $K_i = K(x_i, 2(\epsilon/\pi)^{1/2})$ the circle of radius $2(\epsilon/\pi)^{1/2}$ ($< 1/2$) whose center is at x_i , then at least one quadrant of this circle is contained in \bar{Q} . Since the area of K_i is equal to 4ϵ and since $m^{(2)}(\bar{Q} - S) < \epsilon$ by assumption, each K_i must contain at least one point x'_i which belongs to S , $i = 1, \dots, p-1$.

Let us now put $x'_0 = x_0, x'_p = x_p$. Then $x'_0 = x, x'_1, \dots, x'_p = y$ is a chain of points from S which connects x and y . Consequently,

$$\begin{aligned} d^{(2)}(x, y; S) &\leq \sum_{i=1}^p (d(x'_{i-1}, x'_i))^2 \\ &\leq \sum_{i=1}^p (d(x'_{i-1}, x_{i-1}) + d(x_{i-1}, x_i) + d(x_i, x'_i))^2 \\ (4.4) \quad &\leq p \left(2(\epsilon/\pi)^{1/2} + \frac{1}{p} d(x, y) + 2(\epsilon/\pi)^{1/2} \right)^2 \\ &\leq p \left(4(\epsilon/\pi)^{1/2} + \frac{2^{1/2}}{p} \right)^2 \\ &\leq \left(1 + \frac{1}{\epsilon^{1/2}} \right) (4(\epsilon/\pi)^{1/2} + (2\epsilon)^{1/2})^2 \\ &\leq \epsilon^{1/2} (1 + \epsilon^{1/2}) \left(\frac{4}{\pi^{1/2}} + 2^{1/2} \right)^2 \leq 50\epsilon^{1/2}. \end{aligned}$$

THEOREM 1. If S is a compact subset of a two-dimensional Euclidean space, then the quadratic variation of S is zero, i.e. for any $\eta > 0$, there exists a decomposition $S = \bigcup_{i=1}^p S_i$ of S into a finite number of closed subsets S_i , $i = 1, \dots, p$, such that

$$(4.5) \quad v^{(2)}(S) \leq \sum_{i=1}^p \delta^{(2)}(S_i) < \eta.$$

PROOF. Because of Lemma 4.1 it suffices to discuss the case when S is a subset of the closed unit square $\bar{Q} = \{(u, v) \mid 0 \leq u, v \leq 1\}$ of R^2 . For any positive integer n , let us put

$$(4.6) \quad Q_{k,l}^{(n)} = \left\{ (u, v) \mid \frac{k-1}{2^n} < u < \frac{k}{2^n}, \frac{l-1}{2^n} < v < \frac{l}{2^n} \right\}$$

$k, l = 1, 2, \dots, 2^n$, and

$$(4.7) \quad S_{k,l}^{(n)} = S \cap \bar{Q}_{k,l}^{(n)}$$

where $\bar{Q}_{k,l}^{(n)}$ is the closure of $Q_{k,l}^{(n)}$, $k, l = 1, \dots, 2^n$. Then it is clear that

$$(4.8) \quad S = \bigcup_{k,l=1}^{2^n} S_{k,l}^{(n)} = \bigcup_{(k,l) \in \Delta^{(n)}} S_{k,l}^{(n)}$$

where $\Delta^{(n)}$ denotes the set of all pairs of integers (k, l) , $k, l = 1, \dots, 2^n$.

Let $\epsilon > 0$ be a positive number such that $4\epsilon + 50\epsilon^{1/2} < \eta$, and let us divide the set $\Delta^{(n)}$ into three classes $\Delta_1^{(n)}, \Delta_2^{(n)}, \Delta_3^{(n)}$, by the following conditions:

$$(4.9) \quad \Delta_1^{(n)} = \{ (k, l) \mid m^{(2)}(S_{k,l}^{(n)}) < \epsilon m^{(2)}(\bar{Q}_{k,l}^{(n)}) \},$$

$$(4.10) \quad \Delta_2^{(n)} = \{ (k, l) \mid \epsilon \cdot m^{(2)}(\bar{Q}_{k,l}^{(n)}) \leq m^{(2)}(S_{k,l}^{(n)}) \leq (1 - \epsilon)m^{(2)}(\bar{Q}_{k,l}^{(n)}) \},$$

$$(4.11) \quad \Delta_3^{(n)} = \{ (k, l) \mid (1 - \epsilon)m^{(2)}(\bar{Q}_{k,l}^{(n)}) < m^{(2)}(S_{k,l}^{(n)}) \}.$$

Thus, from the theory of Lebesgue measure in $R^{(2)}$, it follows that, for any fixed $\epsilon > 0$, there exists an integer $n_0(\epsilon)$ such that

$$(4.12) \quad |\Delta_2^{(n)}| < \epsilon \cdot 2^{-2n}$$

for any $n > n_0(\epsilon)$, where $|\Delta_2^{(n)}|$ denotes the number of pairs of (k, l) contained in $\Delta_2^{(n)}$.

From Lemma 4.2 it follows that, for any $(k, l) \in \Delta_1^{(n)}$, there exists a decomposition

$$(4.13) \quad S_{k,l}^{(n)} = \bigcup_{i=1}^{p_{k,l}} S_{k,l,i}^{(n)}$$

of $S_{k,l}^{(n)}$ into a finite number of closed subsets $S_{k,l,i}^{(n)}$, $i = 1, \dots, p_{k,l}$, such that

$$(4.14) \quad \sum_{i=1}^{pk,l} \delta^{(2)}(S_{k,l,i}^{(n)}) < 2m(S_{k,l}^{(n)}) < 2\epsilon 2^{-2n}.$$

Further, from Lemmas 4.1 and 4.3 follows that

$$(4.15) \quad \delta^{(2)}(S_{k,l}^{(n)}) < 50\epsilon^{1/2} \cdot 2^{-2n}$$

for any $(k, l) \in \Delta_3^{(n)}$. If we now consider the decomposition

$$(4.16) \quad S = \left(\bigcup_{(k,l) \in \Delta_1^{(n)}} \bigcup_{i=1}^{pk,l} S_{k,l,i}^{(n)} \right) \cup \left(\bigcup_{(k,l) \in \Delta_2^{(n)}} S_{k,l}^{(n)} \right) \cup \left(\bigcup_{(k,l) \in \Delta_3^{(n)}} S_{k,l}^{(n)} \right),$$

then we see that

$$(4.17) \quad \begin{aligned} v^{(2)}(S) &\leq \sum_{(k,l) \in \Delta_1^{(n)}} \sum_{i=1}^{pk,l} \delta^{(2)}(S_{k,l,i}^{(n)}) \\ &+ \sum_{(k,l) \in \Delta_2^{(n)}} \delta^{(2)}(S_{k,l}^{(n)}) + \sum_{(k,l) \in \Delta_3^{(n)}} \delta^{(2)}(S_{k,l}^{(n)}) \\ &\leq |\Delta_1^{(n)}| \cdot 2\epsilon \cdot 2^{-2n} + |\Delta_2^{(n)}| \cdot 2 \cdot 2^{-2n} \\ &+ |\Delta_3^{(n)}| \cdot 50 \cdot \epsilon^{1/2} \cdot 2^{-2n}, \end{aligned}$$

where $|\Delta_i^{(n)}|$ is the number of pairs (k, l) contained in $\Delta_i^{(n)}$, $i = 1, 2, 3$. Since $|\Delta_1^{(n)}| \leq 2^{2n}$, $|\Delta_3^{(n)}| \leq 2^{2n}$ for all n and $|\Delta_2^{(n)}| < \epsilon \cdot 2^{2n}$ for $n > n_0(\epsilon)$, it follows that the right-hand side of (4.17) is smaller than $2\epsilon + 2\epsilon + 50\epsilon^{1/2} < \eta$ if $n > n_0(\epsilon)$. This proves Theorem 1.

THEOREM 2. *If S is a compact connected subset of $R^{(2)}$, then $\delta^{(2)}(S) = 0$.*

PROOF. This follows from Theorem 1 and the relation (2.8) of Lemma 2.2.

5. Functions satisfying the conditions of Lipschitz of type $(2, M)$. Let $R = \{x, y, \dots\}$ be a metric space with a metric $d(x, y)$. Let α, M be two positive numbers. A real-valued function $f(x)$ defined on a subset S of R is said to satisfy the condition of Lipschitz of type (α, M) on S if

$$(5.1) \quad |f(x) - f(y)| \leq M(d(x, y))^\alpha$$

for any $x, y \in S$. We are mainly interested in the case $\alpha = 1, 2$. (In case $\alpha = 1$, our condition is reduced to the usual Lipschitz condition.)

For any real-valued function $f(x)$ defined on S , we denote by $f(S)$ the set of values taken by $f(x)$ on S . The diameter $\delta(f(S))$ of $f(S)$

is nothing but the oscillation $\sup_{x \in S} f(x) - \inf_{x \in S} f(x)$ of $f(x)$ on S .

LEMMA 5.1. *Let $f(x)$ be a real-valued function defined on a subset S of a metric space R which satisfies the condition of Lipschitz of type $(2, M)$ on S . Then*

$$(5.2) \quad \delta(f(S)) \leq M \cdot \delta^{(2)}(S).$$

PROOF. It suffices to show that

$$(5.3) \quad |f(x) - f(y)| \leq M \cdot d^{(2)}(x, y; S)$$

for all $x, y \in S$, and this follows immediately from the fact that

$$(5.4) \quad \begin{aligned} |f(x) - f(y)| &\leq \sum_{i=1}^p |f(x_{i-1}) - f(x_i)| \\ &\leq M \sum_{i=1}^p (d(x_{i-1}, x_i))^2 \end{aligned}$$

for any chain $x_0 = x, x_1, \dots, x_p = y$ from S .

LEMMA 5.2. *Under the same assumptions as in Lemma 5.1, we have*

$$(5.5) \quad m^{(1)}(f(S)^-) \leq M v^{(2)}(S),$$

where $m^{(1)}(f(S)^-)$ denotes the one-dimensional Lebesgue measure of the closure $f(S)^-$ of $f(S)$. In particular, $f(S)^-$ is a set of Lebesgue measure zero if $v^{(2)}(S) = 0$.

PROOF. For any $\epsilon > 0$, let $S = \bigcup_{i=1}^p S_i$ be a decomposition of S into a finite number of closed subsets $S_i, i = 1, \dots, p$, such that

$$(5.6) \quad \sum_{i=1}^p \delta^{(2)}(S_i) < v^{(2)}(S) + \epsilon.$$

It is then easy to see that

$$(5.7) \quad \begin{aligned} m^{(1)}(f(S)^-) &\leq \sum_{i=1}^p \delta(f(S_i)) \\ &\leq M \sum_{i=1}^p \delta^{(2)}(S_i) \\ &< M(v^{(2)}(S) + \epsilon), \end{aligned}$$

which obviously implies (5.5) since $\epsilon > 0$ is arbitrary.

THEOREM 3. *Let $f(x)$ be a real-valued function defined on a compact subset S of R^2 which satisfies the condition of Lipschitz of type $(2, M)$ on*

S. Then the set $f(S)$ of values taken by $f(x)$ on S is a set of one-dimensional Lebesgue measure zero. Further, $f(x)$ is constant on every component of S . In particular, $f(x)$ is a constant if S is connected.

PROOF. The proof follows immediately from Theorems 1, 2 and Lemmas 5.1, 5.2.

It is easy to see that, for any subset S of a metric space R with $\delta^{(2)}(S) > 0$, there exists a nonconstant function $f(x)$ defined on it which satisfies the condition of Lipschitz of type (2.1) on S . In fact, for any fixed $x_0 \in S$, the function $f(x) = d^{(2)}(x, x_0; S)$ is a nonconstant function defined on S and satisfies

$$(5.8) \quad \begin{aligned} |f(x) - f(y)| &= |d^{(2)}(x, x_0; S) - d^{(2)}(y, x_0; S)| \\ &\leq d^{(2)}(x, y; S) \leq (d(x, y))^2. \end{aligned}$$

In particular, if $S = \{x_t | 0 \leq t \leq 1\}$ is a curve in a Hilbert space or a Euclidean space R^n ($n \geq 3$) discussed in §3 with the property that $d^{(2)}(x_s, x_t; S) > 0$ for any s, t with $0 \leq s < t \leq 1$, then $f(x_t) = t$ is a nonconstant function which satisfies the condition of Lipschitz of type (2.1) on S .

6. Application to a problem in analysis. Let $Q = \{x = (u, v) | 0 < u, v < 1\}$ be the open unit square in R^2 , and let $f(x) = f(u, v)$ be a real-valued function defined on Q . $f(x) = f(u, v)$ is said to belong to the class C^1 if all first partial derivatives of $f(u, v)$ exist and are continuous on Q . $f(x) = f(u, v)$ is said to belong to the class $C^1(\alpha, M)$ if further all first partial derivatives of $f(y, v)$ satisfy the Lipschitz condition of type (α, M) . We are mainly interested in the case $\alpha = 1$.

Let $f(x) = f(u, v)$ be a function belonging to the class C^1 . A point $x = (u, v) \in Q$ is called a *critical point* of f if all first partial derivatives vanish at $x = (u, v)$. The set $S(f)$ of all critical points of f is called the *critical set* of f . It is clear that $S(f)$ is a relatively closed subset of Q , and hence is an F_σ -set.

LEMMA 6.1. *If $f(x)$ belongs to $C^1(1, M)$, then $f(x)$ satisfies the condition of Lipschitz of type $(2, M)$ on its critical set.*

PROOF. The proof is easy and so is omitted.

THEOREM 4. *Let $f(x) = f(u, v)$ be a real-valued function defined on the open unit square $Q = \{x = (u, v) | 0 < u, v < 1\}$ of R^2 which belongs to the class $C^1(1, M)$. Then the set $f(S(f))$ of values taken by $f(x)$ on its critical set $S(f)$ is a set of one-dimensional Lebesgue measure zero. Further, $f(x)$ is constant on every component of $S(f)$.*

PROOF. The proof follows easily from Theorem 3 and Lemma 6.1.

REMARK. It was shown by H. Whitney [4] that there exists a function $f(x) = f(u, v)$ defined on the open unit square Q of R^2 with continuous first partial derivatives which is not constant on a connected part of its critical set $S(f)$. This fact shows that the continuity of the first partial derivatives is not sufficient for Theorem 4, and that something like the condition of Lipschitz of type $(1, M)$ for the first partial derivatives of $f(u, v)$ is necessary for Theorem 4.

If, on the other hand, all second partial derivatives of $f(u, v)$ exist and are continuous, then the proof of Theorem 4 becomes much easier. In fact, in this case it is easy to see that, for any $x \in S(f)$, there exists a neighborhood $V(x)$ of x such that $f(x)$ satisfies the condition of Lipschitz of type $(2, \epsilon)$ on $S(f) \cap V(x)$. From this follows that, for any $\epsilon > 0$, there exists a sequence of squares Q_i , $i = 1, 2, \dots$, such that (i) $\bigcup_{i=1}^{\infty} Q_i = Q$, (ii) $\sum_{i=1}^{\infty} m^{(2)}(Q_i) < m^{(2)}(Q) + \epsilon = 1 + \epsilon$, and (iii) $f(x)$ satisfies the condition of Lipschitz of type $(2, \epsilon)$ on each $S_i = S(f) \cap \bar{Q}_i$, $i = 1, 2, \dots$. From Lemma 6.1 then follows

$$\begin{aligned}
 m^{(1)}(f(S)) &\leq \sum_{i=1}^{\infty} m^{(1)}(f(S_i)) \\
 (6.1) \quad &\leq \sum_{i=1}^{\infty} \delta(f(S_i)) \leq \epsilon \sum_{i=1}^{\infty} \delta^{(2)}(S_i) \\
 &\leq \epsilon \sum_{i=1}^{\infty} \delta^{(2)}(Q_i) \leq 2\epsilon \sum_{i=1}^{\infty} m^{(2)}(Q_i) \\
 &< 2\epsilon(1 + \epsilon).
 \end{aligned}$$

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