QUADRATIC DIAMETER OF A METRIC SPACE AND ITS APPLICATION TO A PROBLEM IN ANALYSIS

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1. **Introduction.** The purpose of this note is to introduce the notion of quadratic diameter and to apply it to a problem in analysis concerning the measure of the set of values taken by a real-valued differentiable function on its critical set. It was proved by A. P. Morse [**3**] that if $f(x)$ is a real-valued function defined on an open set of an n-dimensional Euclidean space which is n-times continuously partially differentiable, then the set of values taken by $f(x)$ on its critical set (i.e. the set of all x at which all first partial derivatives of $f(x)$ vanish) is a set of one-dimensional Lebesgue measure zero. We shall give a simple proof of this result of Morse for the case $n=2$, under a weaker assumption that the first partial derivatives of $f(x)$ satisfy the Lipschitz condition. (It is mentioned in [**3**] that his condition can be weakened and that it is sufficient to assume that all $(n-1)$th partial derivatives of $f(x)$ are totally differentiable; but this weakened condition is still stronger than our condition for the case $n=2$.) It is, however, to be observed that our method seems to work only for the case $n=2$, and the problem of discussing the possibility of similar approach for the higher-dimensional cases $n>2$ is still open.

2. **Quadratic diameter.** Let $R=\{x, y, \cdots\}$ be a metric space with the metric $d(x, y)$. For any subset S of R and for any two points x, y of S, the *quadratic distance* $d^{(2)}(x, y; S)$ of x, y with respect to S is defined by

$$d^{(2)}(x, y, S) = \inf \sum_{i=1}^{p} (d(x_{i-1}, x_i))^2, \tag{2.1}$$

where inf is taken for all possible finite *chains* $x_0=x, x_1, \cdots, x_p=y$ from S. (The *length* p of this chain is also arbitrary.) Further, the *quadratic diameter* $\delta^{(2)}(S)$ of S is defined by

$$\delta^{(2)}(S) = \sup_{x, y\in S} d^{(2)}(x, y; S). \tag{2.2}$$

Finally, the *quadratic variation* $v^{(2)}(S)$ of S is defined by

$$v^{(2)}(S) = \inf \sum_{i=1}^{p} \delta^{(2)}(S_i), \tag{2.3}$$

Received by the editors December 7, 1951.

where inf is taken for all possible decompositions $S=\bigcup_{i=1}^{p}S_i$ of S into a finite number of (not necessarily disjoint) closed subsets S_i, $i=1, \cdots, p$.

We begin with simple observations concerning the quantities defined above.

LEMMA 2.1.

$$d^{(2)}(x, y; S) = d^{(2)}(y, x; S) \leqq (d(x, y))^2, \tag{2.4}$$

$$d^{(2)}(x, z; S) \leqq d^{(2)}(x, y; S) + d^{(2)}(y, z; S) \tag{2.5}$$

for any $x, y, z \in S$.

LEMMA 2.2.

$$v^{(2)}(S) \leqq \delta^{(2)}(S) \leqq (\delta(S))^2, \tag{2.6}$$

where $\delta(S)=\sup_{x,y\in S} d(x, y)$ *is the ordinary diameter of* S. *Further*

$$\delta^{(2)}(S_1 \cup S_2) \leqq \delta^{(2)}(S_1) + \delta^{(2)}(S_2) \tag{2.7}$$

if S_1 *and* S_2 *are not disjoint, and*

$$v^{(2)}(S) = \delta^{(2)}(S) \tag{2.8}$$

if S *is connected.*

The proofs of these lemmas are easy and so omitted.

Because of Lemma 2.1, $d^{(2)}(x, y; S)$ may be considered as a quasi-metric on S. It is possible that $d^{(2)}(x, y; S)=0$ for two different points x, y of S (as will be shown in Lemma 2.3). In this case x, y are called $d^{(2)}$*-equivalent*. It is easy to see that this relation of $d^{(2)}$-equivalence is symmetric and transitive, and that for any $x\in S$ the set $E(x)$ of all $y\in S$ which is $d^{(2)}$-equivalent with x is a closed subset of S. Further, $d^{(2)}(E(x), E(y))=d^{(2)}(x, y; S)$ may be considered as a metric defined on the class of all equivalence classes $E(x)$ of S.

It is to be noticed that $S_1\subset S_2\subset R$ and $x, y\in S_1$ imply

$$d^{(2)}(x, y; S_1) \geqq d^{(2)}(x, y; S_2), \tag{2.9}$$

and that it is possible that the inequality holds in (2.9). Thus the quadratic diameter $\delta^{(2)}(S)$ of S is not necessarily a monotone set function of S.

We do not need the following Lemmas 2.3, 2.4 in the proof of our main result in §4. But a similar argument plays an important role in the proof of Lemma 4.3.

LEMMA 2.3. *If* S *is a subset of an* n*-dimensional Euclidean space* R^n $(n\geqq 1)$, *and if* x, y *are two points of* S *such that the segment* $[x, y]$

connecting x and y is contained in S, then $d^{(2)}(x, y; S)=0$.

PROOF. For any integer $p\geqq 1$, let $x_0=x, x_1, \cdots, x_p=y$ be a chain of points from S which divide the segment $[x, y]$ into p equal parts. Then $\sum_{i=1}^{p} (d(x_{i-1}, x_i))^2=(1/p)(d(x, y))^2$, and this implies that $d^{(2)}(x, y; S)=0$.

COROLLARY. *If S is a convex subset of a Euclidean space R^n $(n\geqq 1)$, then $\delta^{(2)}(S)=0$.*

LEMMA 2.4. *If x, y are two points of a subset S of a metric space R which can be connected by a rectifiable curve in S, then $d^{(2)}(x, y; S)=0$.*

PROOF. Let $C=\{x_t \mid 0\leqq t\leqq 1\}$ $(x_0=x, x_1=y)$ be a rectifiable curve which connects x and y in S. Let l be the curve length of C. Then, for any $\epsilon>0$, there exist real numbers $t_0, t_1, \cdots, t_p$ such that $t_0=0<t_1<\cdots<t_p=1$, $d(x_{i-1}, x_i)<\epsilon$, $i=1, \cdots, p$, and

$$(2.10) \qquad \sum_{i=1}^{p} d(x_{t_{i-1}}, x_{t_i}) < l+\epsilon.$$

From this follows

$$(2.11) \qquad \sum_{i=1}^{p} (d(x_{t_{i-1}}, x_{t_i}))^2 < \epsilon(l+\epsilon)$$

and this implies $d^{(2)}(x, y; S)=0$.

COROLLARY. *If S is a subset of a metric space R such that any two points x, y of S can be connected by a rectifiable curve in S, then $\delta^{(2)}(S)=0$.*

The question then arises to determine, given a metric space R, whether or not there exists a connected subset S of R with the property that $d^{(2)}(x, y; S)>0$ for any two different points x, y of S. We shall show in §3 that in a Hilbert space or in a Euclidean space R^n with $n\geqq 3$ there exists a simple curve $S=\{x_t \mid 0\leqq t\leqq 1\}$ such that $d^{(2)}(x_s, x_t; S)>0$ for any s, t with $0\leqq s<t\leqq 1$.

3. Examples of continuous curves with a positive quadratic diameter. Let $S=\{x_t \mid 0\leqq t\leqq 1\}$ be a curve in a Hilbert space R with the following properties: (i) the vectors x_t-x_s and $x_{t'}-x_{s'}$ are orthogonal if $0\leqq s<t\leqq s'<t'\leqq 1$, (ii) $d(x_s, x_t)=(t-s)^{1/2}$ if $0\leqq s<t\leqq 1$. Then it is easy to see that

$$(3.1) \qquad \sum_{i=1}^{p} (d(x_{t_{i-1}}, x_{t_i}))^2 = (d(x_s, x_t))^2 = t-s$$

for any $s, t_0, t_1, \cdots, t_p, t$ such that $0 \leqq s = t_0 < t_1 < \cdots < t_p = t \leqq 1$. From this follows that

$$d^{(2)}(x_s, x_t; S) = t - s \tag{3.2}$$

for any s, t with $0 \leqq s < t \leqq 1$. An example of a curve S in a Hilbert space with the properties (i), (ii) can be given as follows: Let $R = L^2(0, 1)$ be the L^2-space of all real-valued square integrable functions $x(u)$ defined on the unit interval $I = \{u \mid 0 \leqq u \leqq 1\}$ with the usual norm

$$\|x\| = \left(\int_0^1 |x(u)|^2 du\right)^{1/2} < \infty, \tag{3.3}$$

and let $S = \{x_t \mid 0 \leqq t \leqq 1\}$ be a curve in R defined by

$$x_t(u) = \begin{cases} 1, & 0 \leqq u \leqq t, \\ 0, & t < u \leqq 1. \end{cases} \tag{3.4}$$

Then it is easy to see that S has the properties (i), (ii) above. It is to be noticed that from Lemma 2.4 follows that S is not rectifiable. In fact it is easy to see that

$$\sum_{i=1}^{p} d(x_{(i-1)/p}, x_{i/p}) = p \cdot \frac{1}{p^{1/2}} = p^{1/2} \rightarrow \infty$$

as $p \rightarrow \infty$. Such curves have been discussed by N. Wiener [5], A. Kolmogoroff [1], and P. Lévy [2] in connection with the theory of Brownian motion.

In case R is a finite-dimensional Euclidean space R^n ($n \geqq 3$), the existence of a curve $S = \{x_t \mid 0 \leqq t \leqq 1\}$ with the same property can be shown by using a Peano curve. In fact, let $\{(u_t, v_t) \mid 0 \leqq t \leqq 1\}$ be the usual parametrization of the standard Peano curve filling the closed unit square $\overline{Q} = \{(u, v) \mid 0 \leqq u, v \leqq 1\}$ of R^2 with the property that, for any $n = 1, 2, \cdots$ and for any $i = 1, \cdots, 2^{2n}$, the image of the closed interval

$$\overline{I}_i^{(n)} = \{t \mid (i-1)2^{-2n} \leqq t \leqq i \cdot 2^{-2n}\} \tag{3.5}$$

is one of the 2^{2n} closed squares

$$\overline{Q}_{k,l}^{(n)} = \{(u, v) \mid (k-1)2^{-n} \leqq u \leqq k2^{-n}, (l-1)2^{-n} \leqq v \leqq l \cdot 2^{-n}\}, \quad k, l = 1, \cdots, 2^n. \tag{3.6}$$

Then $S = \{x_t = (u_t, v_t, t) \mid 0 \leqq t \leqq 1\}$ is a simple curve in R^3 whose projection on the (u, v)-plane fills up the closed unit interval $\overline{Q} = \{(u, v) | 0 \leqq u, v \leqq 1\}$. It is easy to see that $d^{(2)}(x_s, x_t; S) > 0$ for any s, t with $0 \leqq s < t \leqq 1$.

On the other hand, it is impossible to construct similar examples in R^2. In fact, as will be shown in Theorem 2 below, $\delta^{(2)}(S) = 0$ for any compact connected subset S of R^2 and hence $d^{(2)}(x, y; S) = 0$ for any two points x, y of a continuous curve S in $R^{(2)}$.

If we consider the case when S is compact but not connected, then it is clear that $\delta^{(2)}(S) > 0$. In fact, if $S = S_1 \cup S_2$ is a decomposition of S into two disjoint compact subsets S_1 and S_2, then there exists a constant $\eta > 0$ such that $d(x, y) \geqq \eta > 0$ for any $x \in S_1$, $y \in S_2$ and hence $\delta^{(2)}(S) \geqq d^{(2)}(x, y; S) \geqq \eta^2 > 0$ for any $x \in S_1$, $y \in S_2$. Thus it seems as if there is nothing we can conclude about $\delta^{(2)}(S)$ in general. But we can show that if S is a compact subset of R^2, then the quadratic variation $v^2(S)$ of S is zero, or in other words, that, for any $\eta > 0$, there exists a decomposition $S = \bigcup_{i=1}^{p} S_i$ of S into a finite number of subsets S_i, $i = 1, \cdots, p$, such that $\sum_{i=1}^{p} \delta^{(2)}(S_i) < \eta$. This will be proved in Theorem 1.

4. Quadratic diameter of a set in R^2.

LEMMA 4.1. *Let S be a compact subset of a two-dimensional Euclidean space R^2. If we denote by αS a subset of R^2 which is homothetic with S with the ratio α:1, then*

$$\delta^{(2)}(\alpha S) = \alpha^2 \delta^{(2)}(S). \tag{4.1}$$

PROOF. The proof is clear from definition.

LEMMA 4.2. *Let S be a compact subset of the closed unit square $\overline{Q} = \{(u, v) | 0 \leqq u, v \leqq 1\}$ of R^2 with $m^{(2)}(S) < \epsilon$, where $m^{(2)}(S)$ denotes the two-dimensional Lebesgue measure of S. Then there exists a decomposition $S = \bigcup_{i=1}^{p} S_i$ of S into a finite number of closed subsets S_i, $i = 1, \cdots, p$, such that*

$$\sum_{i=1}^{p} \delta^{(2)}(S_i) < 2\epsilon. \tag{4.2}$$

PROOF. Since $m^{(2)}(S) < \epsilon$, there exists an open set O of R^2 such that $S \subset O$ and $m^{(2)}(O) < \epsilon$. Since O is open, there exists a countable number of open squares Q_i, $i = 1, 2, \cdots$, such that $O = \bigcup_{i=1}^{\infty} Q_i$ and $\sum_{i=1}^{\infty} m^{(2)}(Q_i) < \epsilon$. Since S is compact, there exists a positive integer p such that $S \subset \bigcup_{i=1}^{p} Q_i$. If we put $S_i = S \cap \overline{Q}_i$ where $\overline{Q}_i$ is the closure of Q_i, $i = 1, \cdots, p$, then $S = \bigcup_{i=1}^{p} S_i$ is a required decomposition. In fact, in the notation of (2.6), it is clear that $\sum_{i=1}^{p} \delta^{(2)}(S_i)$

$\leqq \sum_{i=1}^{p} (\delta(S_i))^2 \leqq \sum_{i=1}^{p} (\delta(\overline{Q}_i))^2 = 2\sum_{i=1}^{p} m^{(2)}(Q_i) < 2\sum_{i=1}^{\infty} m^{(2)}(Q_i)$ $< 2\epsilon$.

LEMMA 4.3. *Let S be a compact subset of the closed unit square $\overline{Q} = \{(u, v) \mid 0 \leqq u, v \leqq 1\}$ of R^2 such that $m^{(2)}(\overline{Q} - S) < \epsilon$, where $\epsilon < \pi/16$ and $m^{(2)}(\overline{Q} - S)$ denotes the two-dimensional Lebesgue measure of $\overline{Q} - S$. Then*

$$\delta^{(2)}(S) < 50\,\epsilon^{1/2}. \tag{4.3}$$

PROOF. Let p be a positive integer such that $p^{-1} \leqq \epsilon^{1/2} < (p-1)^{-1}$. For any two points x, y of S, let $x_0 = x, x_1, \cdots, x_p = y$ be the chain of points from $\overline{Q}$ of length p which divides the segment $[x, y]$ in p equal parts. We do not know whether these dividing points $x_1, \cdots, x_{p-1}$ belong to S or not. But it is possible to show that for each $i = 1, \cdots, p-1$, there exists a point x_i' from S such that $d(x_i, x_i') < 2(\epsilon/\pi)^{1/2}$. In fact, if we denote by $K_i = K(x_i, 2(\epsilon/\pi)^{1/2})$ the circle of radius $2(\epsilon/\pi)^{1/2}$ $(<1/2)$ whose center is at x_i, then at least one quadrant of this circle is contained in $\overline{Q}$. Since the area of K_i is equal to 4ϵ and since $m^{(2)}(\overline{Q} - S) < \epsilon$ by assumption, each K_i must contain at least one point x_i' which belongs to S, $i = 1, \cdots, p-1$.

Let us now put $x_0' = x_0$, $x_p' = x_p$. Then $x_0' = x, x_1', \cdots, x_p' = y$ is a chain of points from S which connects x and y. Consequently,

$$\begin{aligned} d^{(2)}(x, y; S) &\leqq \sum_{i=1}^{p} (d(x_{i-1}', x_i'))^2 \\ &\leqq \sum_{i=1}^{p} (d(x_{i-1}', x_{i-1}) + d(x_{i-1}, x_i) + d(x_i, x_i'))^2 \\ &\leqq p\left(2(\epsilon/\pi)^{1/2} + \frac{1}{p} d(x, y) + 2(\epsilon/\pi)^{1/2}\right)^2 \\ &\leqq p\left(4(\epsilon/\pi)^{1/2} + \frac{2^{1/2}}{p}\right)^2 \\ &\leqq \left(1 + \frac{1}{\epsilon^{1/2}}\right)(4(\epsilon/\pi)^{1/2} + (2\epsilon)^{1/2})^2 \\ &\leqq \epsilon^{1/2}(1 + \epsilon^{1/2})\left(\frac{4}{\pi^{1/2}} + 2^{1/2}\right)^2 \leqq 50\epsilon^{1/2}. \end{aligned} \tag{4.4}$$

THEOREM 1. *If S is a compact subset of a two-dimensional Euclidean space, then the quadratic variation of S is zero, i.e. for any $\eta > 0$, there exists a decomposition $S = \bigcup_{i=1}^{p} S_i$ of S into a finite number of closed subsets S_i, $i = 1, \cdots, p$, such that*

$$(4.5) \qquad v^{(2)}(S) \leqq \sum_{i=1}^{p} \delta^{(2)}(S_i) < \eta.$$

PROOF. Because of Lemma 4.1 it suffices to discuss the case when S is a subset of the closed unit square $\overline{Q} = \{(u, v) \mid 0 \leqq u, v \leqq 1\}$ of R^2. For any positive integer n, let us put

$$(4.6) \qquad Q_{k,l}^{(n)} = \left\{(u, v) \,\middle|\, \frac{k-1}{2^n} < u < \frac{k}{2^n}, \ \frac{l-1}{2^n} < v < \frac{l}{2^n}\right\}$$

$k, l = 1, 2, \cdots, 2^n$, and

$$(4.7) \qquad S_{k,l}^{(n)} = S \cap \overline{Q}_{k,l}^{(n)},$$

where $\overline{Q}_{k,l}^{(n)}$ is the closure of $Q_{k,l}^{(n)}$, $k, l = 1, \cdots, 2^n$. Then it is clear that

$$(4.8) \qquad S = \bigcup_{k,l=1}^{2n} S_{k,l}^{(n)} = \bigcup_{(k,l) \in \Delta^{(n)}} S_{k,l}^{(n)},$$

where $\Delta^{(n)}$ denotes the set of all pairs of integers (k, l), $k, l = 1, \cdots, 2^n$.

Let $\epsilon > 0$ be a positive number such that $4\epsilon + 50\epsilon^{1/2} < \eta$, and let us divide the set $\Delta^{(n)}$ into three classes $\Delta_1^{(n)}$, $\Delta_2^{(n)}$, $\Delta_3^{(n)}$, by the following conditions:

$$(4.9) \qquad \Delta_1^{(n)} = \{(k, l) \mid m^{(2)}(S_{k,l}^{(n)}) < \epsilon m^{(2)}(\overline{Q}_{k,l}^{(n)})\},$$

$$(4.10) \qquad \Delta_2^{(n)} = \{(k, l) \mid \epsilon \cdot m^{(2)}(\overline{Q}_{k,l}^{(n)}) \leqq m^{(2)}(S_{k,l}^{(n)}) \leqq (1-\epsilon) m^{(2)}(\overline{Q}_{k,l}^{(n)})\},$$

$$(4.11) \qquad \Delta_3^{(n)} = \{(k, l) \mid (1-\epsilon) m^{(2)}(\overline{Q}_{k,l}^{(n)}) < m^{(2)}(S_{k,l}^{(n)})\}.$$

Thus, from the theory of Lebesgue measure in $R^{(2)}$, it follows that, for any fixed $\epsilon > 0$, there exists an integer $n_0(\epsilon)$ such that

$$(4.12) \qquad |\Delta_2^{(n)}| < \epsilon \cdot 2^{-2n}$$

for any $n > n_0(\epsilon)$, where $|\Delta_2^{(n)}|$ denotes the number of pairs of (k, l) contained in $\Delta_2^{(n)}$.

From Lemma 4.2 it follows that, for any $(k, l) \in \Delta_1^{(n)}$, there exists a decomposition

$$(4.13) \qquad S_{k,l}^{(n)} = \bigcup_{i=1}^{p_{k,l}} S_{k,l,i}^{(n)}$$

of $S_{k,l}^{(n)}$ into a finite number of closed subsets $S_{k,l,i}^{(n)}$, $i = 1, \cdots, p_{k,l}$, such that

$$\sum_{i=1}^{p_{k,l}} \delta^{(2)}(S_{k,l,i}^{(n)}) < 2m(S_{k,l}^{(n)}) < 2\epsilon 2^{-2n}. \tag{4.14}$$

Further, from Lemmas 4.1 and 4.3 follows that

$$\delta^{(2)}(S_{k,l}^{(n)}) < 50\epsilon^{1/2}\cdot 2^{-2n} \tag{4.15}$$

for any $(k, l) \in \Delta_3^{(n)}$. If we now consider the decomposition

$$\begin{aligned} S = &\left(\bigcup_{(k,l)\in\Delta_1^{(n)}} \bigcup_{i=1}^{p_{k,l}} S_{k,l,i}^{(n)} \right) \\ &\cup \left(\bigcup_{(k,l)\in\Delta_2^{(n)}} S_{k,l}^{(n)} \right) \cup \left(\bigcup_{(k,l)\in\Delta_3^{(n)}} S_{k,l}^{(n)} \right), \end{aligned} \tag{4.16}$$

then we see that

$$\begin{aligned} v^{(2)}(S) \leqq &\sum_{(k,l)\in\Delta_1^{(n)}} \sum_{i=1}^{p_{k,l}} \delta^{(2)}(S_{k,l,i}^{(n)}) \\ &+ \sum_{(k,l)\in\Delta_2^{(n)}} \delta^{(2)}(S_{k,l}^{(n)}) + \sum_{(k,l)\in\Delta_3^{(n)}} \delta^{(2)}(S_{k,l}^{(n)}) \\ \leqq &\left| \Delta_1^{(n)} \right| \cdot 2\epsilon\cdot 2^{-2n} + \left| \Delta_2^{(n)} \right| \cdot 2\cdot 2^{-2n} \\ &+ \left| \Delta_3^{(n)} \right| \cdot 50\cdot \epsilon^{1/2}\cdot 2^{-2n}, \end{aligned} \tag{4.17}$$

where $|\Delta_i^{(n)}|$ is the number of pairs (k, l) contained in $\Delta_i^{(n)}$, $i = 1, 2, 3$. Since $|\Delta_1^{(n)}| \leqq 2^{2n}$, $|\Delta_3^{(n)}| \leqq 2^{2n}$ for all n and $|\Delta_2^{(n)}| < \epsilon\cdot 2^{2n}$ for $n > n_0(\epsilon)$, it follows that the right-hand side of (4.17) is smaller than $2\epsilon + 2\epsilon + 50\epsilon^{1/2} < \eta$ if $n > n_0(\epsilon)$. This proves Theorem 1.

THEOREM 2. *If S is a compact connected subset of $R^{(2)}$, then $\delta^{(2)}(S) = 0$.*

PROOF. This follows from Theorem 1 and the relation (2.8) of Lemma 2.2.

5. **Functions satisfying the conditions of Lipschitz of type** $(2, M)$. Let $R = \{x, y, \cdots\}$ be a metric space with a metric $d(x, y)$. Let α, M be two positive numbers. A real-valued function $f(x)$ defined on a subset S of R is said to satisfy the condition of Lipschitz of type (α, M) on S if

$$|f(x) - f(y)| \leqq M(d(x, y))^\alpha \tag{5.1}$$

for any $x, y \in S$. We are mainly interested in the case $\alpha = 1, 2$. (In case $\alpha = 1$, our condition is reduced to the usual Lipschitz condition.)

For any real-valued function $f(x)$ defined on S, we denote by $f(S)$ the set of values taken by $f(x)$ on S. The diameter $\delta(f(S))$ of $f(S)$

is nothing but the oscillation $\sup_{x\in S} f(x)-\inf_{x\in S} f(x)$ of $f(x)$ on S.

LEMMA 5.1. *Let $f(x)$ be a real-valued function defined on a subset S of a metric space R which satisfies the condition of Lipschitz of type* $(2, M)$ *on S. Then*

$$\delta(f(S)) \leqq M\cdot\delta^{(2)}(S). \tag{5.2}$$

PROOF. It suffices to show that

$$|f(x) - f(y)| \leqq M\cdot d^{(2)}(x, y; S) \tag{5.3}$$

for all $x, y\in S$, and this follows immediately from the fact that

$$\begin{aligned} |f(x) - f(y)| &\leqq \sum_{i=1}^{p} |f(x_{i-1}) - f(x_i)| \\ &\leqq M\sum_{i=1}^{p} (d(x_{i-1}, x_i))^2 \end{aligned} \tag{5.4}$$

for any chain $x_0=x, x_1, \cdots, x_p=y$ from S.

LEMMA 5.2. *Under the same assumptions as in Lemma* 5.1, *we have*

$$m^{(1)}(f(S)^-) \leqq Mv^{(2)}(S), \tag{5.5}$$

where $m^{(1)}(f(S)^-)$ denotes the one-dimensional Lebesgue measure of the closure $f(S)^-$ of $f(S)$. In particular, $f(S)^-$ is a set of Lebesgue measure zero if $v^{(2)}(S)=0$.

PROOF. For any $\epsilon>0$, let $S=\bigcup_{i=1}^{p} S_i$ be a decomposition of S into a finite number of closed subsets S_i, $i=1, \cdots, p$, such that

$$\sum_{i=1}^{p} \delta^{(2)}(S_i) < v^{(2)}(S) + \epsilon. \tag{5.6}$$

It is then easy to see that

$$\begin{aligned} m^{(1)}(f(S)^-) &\leqq \sum_{i=1}^{p} \delta(f(S_i)) \\ &\leqq M\sum_{i=1}^{p} \delta^{(2)}(S_i) \\ &< M(v^{(2)}(S) + \epsilon), \end{aligned} \tag{5.7}$$

which obviously implies (5.5) since $\epsilon>0$ is arbitrary.

THEOREM 3. *Let $f(x)$ be a real-valued function defined on a compact subset S of R^2 which satisfies the condition of Lipschitz of type* $(2, M)$ *on*

S. Then the set $f(S)$ of values taken by $f(x)$ on S is a set of one-dimensional Lebesgue measure zero. Further, $f(x)$ is constant on every component of S. In particular, $f(x)$ is a constant if S is connected.

PROOF. The proof follows immediately from Theorems 1, 2 and Lemmas 5.1, 5.2.

It is easy to see that, for any subset S of a metric space R with $\delta^{(2)}(S)>0$, there exists a nonconstant function $f(x)$ defined on it which satisfies the condition of Lipschitz of type (2.1) on S. In fact, for any fixed $x_0 \in S$, the function $f(x)=d^{(2)}(x, x_0; S)$ is a nonconstant function defined on S and satisfies

$$(5.8)\qquad \begin{aligned} |f(x)-f(y)| &= |d^{(2)}(x, x_0; S) - d^{(2)}(y, x_0; S)| \\ &\leqq d^{(2)}(x, y; S) \leqq (d(x, y))^2. \end{aligned}$$

In particular, if $S=\{x_t \mid 0\leqq t\leqq 1\}$ is a curve in a Hilbert space or a Euclidean space R^n $(n\geqq 3)$ discussed in §3 with the property that $d^{(2)}(x_s, x_t; S)>0$ for any s, t with $0\leqq s<t\leqq 1$, then $f(x_t)=t$ is a nonconstant function which satisfies the condition of Lipschitz of type (2.1) on S.

6. **Application to a problem in analysis.** Let $Q=\{x=(u, v) \mid 0<u, v<1\}$ be the open unit square in R^2, and let $f(x)=f(u, v)$ be a real-valued function defined on Q. $f(x)=f(u, v)$ is said to belong to the class C^1 if all first partial derivatives of $f(u, v)$ exist and are continuous on Q. $f(x)=f(u, v)$ is said to belong to the class $C^1(\alpha, M)$ if further all first partial derivatives of $f(y, v)$ satisfy the Lipschitz condition of type (α, M). We are mainly interested in the case $\alpha=1$.

Let $f(x)=f(u, v)$ be a function belonging to the class C'. A point $x=(u, v)\in Q$ is called a *critical point* of f if all first partial derivatives vanish at $x=(u, v)$. The set $S(f)$ of all critical points of f is called the *critical set* of f. It is clear that $S(f)$ is a relatively closed subset of Q, and hence is an F_σ-set.

LEMMA 6.1. *If $f(x)$ belongs to $C^1(1, M)$, then $f(x)$ satisfies the condition of Lipschitz of type* $(2, M)$ *on its critical set.*

PROOF. The proof is easy and so is omitted.

THEOREM 4. *Let $f(x)=f(u, v)$ be a real-valued function defined on the open unit square $Q=\{x=(u, v) \mid 0<u, v<1\}$ of R^2 which belongs to the class $C'(1, M)$. Then the set $f(S(f))$ of values taken by $f(x)$ on its critical set $S(f)$ is a set of one-dimensional Lebesgue measure zero. Further, $f(x)$ is constant on every component of $S(f)$.*

PROOF. The proof follows easily from Theorem 3 and Lemma 6.1.

REMARK. It was shown by H. Whitney [4] that there exists a function $f(x)=f(u, v)$ defined on the open unit square Q of R^2 with continuous first partial derivatives which is not constant on a connected part of its critical set $S(f)$. This fact shows that the continuity of the first partial derivatives is not sufficient for Theorem 4, and that something like the condition of Lipschitz of type $(1, M)$ for the first partial derivatives of $f(u, v)$ is necessary for Theorem 4.

If, on the other hand, all second partial derivatives of $f(u, v)$ exist and are continuous, then the proof of Theorem 4 becomes much easier. In fact, in this case it is easy to see that, for any $x\in S(f)$, there exists a neighborhood $V(x)$ of x such that $f(x)$ satisfies the condition of Lipschitz of type $(2, \epsilon)$ on $S(f)\cap V(x)$. From this follows that, for any $\epsilon>0$, there exists a sequence of squares Q_i, $i=1, 2, \cdots$, such that (i) $\bigcup_{i=1}^{\infty}Q_i=Q$, (ii) $\sum_{i=1}^{\infty} m^{(2)}(Q_i)<m^{(2)}(Q)+\epsilon=1+\epsilon$, and (iii) $f(x)$ satisfies the condition of Lipschitz of type $(2, \epsilon)$ on each $S_i=S(f)\cap\overline{Q}_i$, $i=1, 2, \cdots$. From Lemma 6.1 then follows

$$
\begin{aligned}
m^{(1)}(f(S)) &\leqq \sum_{i=1}^{\infty} m^{(1)}(f(S_i)) \\
&\leqq \sum_{i=1}^{\infty} \delta(f(S_i)) \leqq \epsilon\sum_{i=1}^{\infty} \delta^{(2)}(S_i) \\
&\leqq \epsilon\sum_{i=1}^{\infty} \delta^{(2)}(Q_i) \leqq 2\epsilon\sum_{i=1}^{\infty} m^{(2)}(Q_i) \\
&< 2\epsilon(1+\epsilon).
\end{aligned}
\tag{6.1}
$$

BIBLIOGRAPHY

1. A. Kolmogoroff, *Wienersche Spiralen und einige interessante Kurven im Hilbertschen Raume*, C. R. (Doklady) Acad. Sci. URSS. N.S. vol. 26 (1940) pp. 115–118.

2. P. Lévy, *Théorie de l'addition des variables aléatoires*, 1937.

3. A. P. Morse, *The behavior of a function on its critical set*, Ann. of Math. vol. 40 (1939) pp. 62–70.

4. H. Whitney, *A function not constant on a set of critical points*, Duke Math. J. vol. 1 (1935) pp. 514–517.

5. N. Wiener, *Generalized harmonic analysis*, Acta Math. vol. 54 (1930) pp. 117–258.

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