

## SOME INEQUALITIES RELATED TO ABEL'S METHOD OF SUMMATION

W. B. PENNINGTON

1. It is well known that if

$$(1) \quad x = e^{-1/u},$$

then there exists a constant  $\rho$  such that

$$(2) \quad \limsup_{u \rightarrow \infty} \left| \sum_{n=0}^{\infty} a_n x^n - \sum_{n \leq u} a_n \right| \leq \rho \limsup_{n \rightarrow \infty} |n a_n|$$

for any series  $\sum a_n$ . This inequality is the source of Tauber's  $o$ -converse of Abel's theorem [Tauber 9]. It is also the source of the following theorem of Vijayaraghavan [10, Theorem 1]:

**THEOREM 1.** *Suppose that the series  $\sum a_n x^n$  is convergent for  $0 < x < 1$ , to the sum  $f(x)$  say, and that, for some fixed real number  $\theta$ ,  $e^{i\theta} f(x) \rightarrow +\infty$  as  $x \rightarrow 1-0$ . Suppose further that  $a_n = O(1/n)$  as  $n \rightarrow \infty$ . Then  $e^{i\theta} \sum_{n \leq u} a_n \rightarrow +\infty$  as  $u \rightarrow \infty$ .*

Theorem 1 may be stated rather less precisely as follows: *If the series  $\sum a_n$  is summable (A) to the sum  $s$  with  $|s| = \infty$ , and if  $a_n = O(1/n)$ , then  $\sum a_n = s$ .* In this form it is seen to be an analogue, for infinite  $s$ , of Littlewood's well known  $O$ -Tauberian theorem for Abel summability [Littlewood 8]. Vijayaraghavan showed that the corresponding analogue of the Hardy-Littlewood "one-sided" Tauberian theorem for Abel summability [Hardy and Littlewood 6] is not true. He proved the following theorem [Vijayaraghavan 10, Theorem 3], and showed by an example that it is "best possible."

**THEOREM 2.** *Suppose that the series  $\sum a_n x^n$  is convergent for  $0 < x < 1$ , to the sum  $f(x)$  say, that  $f(x) \rightarrow -\infty$  as  $x \rightarrow 1-0$ , and that the numbers  $a_n$  are real and satisfy the inequality*

$$a_n > - \frac{K}{n \log \log n}$$

when  $n \geq 3$ . Then

$$\sum_{n \leq u} a_n \rightarrow -\infty \text{ as } u \rightarrow \infty.$$

It is the object of this paper to obtain an inequality related to

---

Received by the editors January 7, 1952.

Theorem 2 in the same way as the inequality (2) is related to Theorem 1.

Recently the inequality (2) and similar ones have received considerable attention [1; 2; 3; 4; 5; 7; 11]. Hartman [7] has found the best possible value  $\bar{p}$  for the constant  $\rho$  in (2) ( $\bar{p} = 1.01598 \dots$ ). Agnew [1; 2; 2 contains an account of the previous work on the subject] has shown that if (1) is replaced by  $x = e^{-u/u}$ , then (2) remains true with  $\rho$  depending on  $q$ , and he has shown that the best possible value  $\bar{p}(q)$  is least when  $q = \log 2$  ( $\bar{p}(\log 2) = 0.96804 \dots$ ).

2. Theorem 2 is clearly a corollary of the theorem:

**THEOREM 3.** *Suppose that the series  $\sum a_n x^n$  is convergent for  $0 < x < 1$ , and that the numbers  $a_n$  are real. Let*

$$x = \exp\left(-\frac{r}{u(\log u)^p}\right),$$

where  $u > 0$ , and  $p$  and  $r$  are any fixed real numbers satisfying  $p \geq 1$  and  $r > 0$ . Then

$$(3) \quad \liminf_{u \rightarrow \infty} \left\{ \sum_{n=0}^{\infty} a_n x^n - x^u \sum_{n \leq u} a_n \right\} \geq p \liminf_{n \rightarrow \infty} a_n n \log \log n,$$

and the factor  $p$  on the right-hand side is the smallest for which the inequality is true.

The theorem is obviously true if the right-hand side is equal to  $-\infty$ . We may therefore suppose that

$$\liminf_{n \rightarrow \infty} a_n n \log \log n = \alpha > -\infty.$$

Let  $\beta < \alpha$ . Then for all but a finite number of the terms  $a_n$  we shall have

$$(4) \quad a_n n \log \log n > \beta.$$

It will be sufficient to show that the left side of (3) is greater than  $p\beta$ .

Since

$$x^u = \exp\left(-\frac{r}{(\log u)^p}\right) \rightarrow 1$$

as  $u \rightarrow \infty$ , we may change a finite number of the terms  $a_n$  without changing the value of the left side of (3), and we shall suppose that  $a_0 = a_1 = a_2 = 0$ , and that (4) holds for  $n \geq 3$ .

Then

$$(5) \quad \sum_{n=0}^{\infty} a_n x^n - x^u \sum_{n \leq u} a_n = \sum_{n \leq u} a_n (x^n - x^u) + \sum_{n > u} a_n x^n$$

$$> \beta \left\{ \sum_{3 \leq n \leq u} \frac{x^n - x^u}{n \log \log n} + \sum_{n > u} \frac{x^n}{n \log \log n} \right\}$$

$$= \beta t(u)$$

say. We have now to show that  $t(u) \rightarrow p$  as  $u \rightarrow \infty$ .

Since  $0 < x < 1$  and  $u(\log u)^p = -r/\log x$ ,

$$(6) \quad t(u) \leq (1 - x^u) \sum_{3 \leq n \leq u} \frac{1}{n \log \log n} + \frac{1}{\log \log u} \sum_{u \leq n \leq u(\log u)^p} \frac{1}{n}$$

$$+ \frac{1}{\log \log u} \sum_{n > -r/\log x} \frac{x^n}{n}.$$

Now

$$1 - x^u = 1 - \exp\left(-\frac{r}{(\log u)^p}\right) = O\left(\frac{1}{(\log u)^p}\right),$$

and

$$\sum_{3 \leq n \leq u} \frac{1}{n \log \log n} < 1 + \int_3^u \frac{dv}{v \log \log v} = O\left(\frac{\log u}{\log \log u}\right),$$

so that the first term on the right side of (6) is

$$O\left(\frac{1}{(\log u)^{p-1} \log \log u}\right) = O\left(\frac{1}{\log \log u}\right)$$

since  $p \geq 1$ . Since  $r > 0$ ,

$$\sum_{n > -r/\log x} \frac{x^n}{n} < 1 + \int_{-r/\log x}^{\infty} x^t \frac{dt}{t} = 1 + \int_1^{\infty} x^{-vr/\log x} \frac{dv}{v}$$

$$= 1 + \int_1^{\infty} e^{-r^2 v^{-1}} dv < \infty,$$

and so the third term on the right side of (6) is  $O(1/\log \log u)$ . It remains to consider the second term. We have

$$\sum_{1 \leq n \leq y} \frac{1}{n} = \log y + O(1)$$

as  $y \rightarrow \infty$ , and so

$$\begin{aligned} \sum_{u < n \leq u(\log u)^p} \frac{1}{n} &= \sum_{n \leq u(\log u)^p} \frac{1}{n} - \sum_{n \leq u} \frac{1}{n} \\ &= \log \{u(\log u)^p\} - \log u + O(1) \\ &= p \log \log u + O(1) \end{aligned}$$

as  $u \rightarrow \infty$ . It follows that the second term is equal to

$$p + O\left(\frac{1}{\log \log u}\right)$$

as  $u \rightarrow \infty$ . Hence, by (6), we have

$$t(u) \leq p + O\left(\frac{1}{\log \log u}\right) = p + o(1)$$

as  $u \rightarrow \infty$ . On the other hand

$$\begin{aligned} t(u) &\geq \sum_{u < n \leq u(\log u)^p / \log \log u} \frac{x^n}{n \log \log n} \\ &\geq \frac{\exp(-1/\log \log u)}{\log \log \{u(\log u)^p / \log \log u\}} \sum_{u < n \leq u(\log u)^p / \log \log u} \frac{1}{n} \\ &= \frac{1 + O(1/\log \log u)}{\log \log u + O(\log \log u / \log u)} \{p \log \log u + O(\log \log \log u)\} \\ &= p + O\left(\frac{\log \log \log u}{\log \log u}\right) \\ &= p + o(1) \end{aligned}$$

as  $u \rightarrow \infty$ . Thus

$$t(u) \rightarrow p \text{ as } u \rightarrow \infty,$$

and (3) is proved.

The example

$$(7) \quad a_n = \begin{cases} 0 & (n = 0, 1, 2), \\ -1/n \log \log n & (n \geq 3) \end{cases}$$

proves that the inequality (5) is the best possible, and, consequently, that  $p$  is the best possible constant in (3).

This completes the proof of Theorem 3.

If we impose another condition on the series  $\sum_0^\infty a_n$ , we can remove the factor  $x^n$  from the left side of (3), and so obtain a closer analogy with (2).

THEOREM 4. *Suppose that the hypotheses of Theorem 3 are fulfilled, and that, in addition, the sum of the series  $\sum_0^\infty a_n x^n$  is bounded above as  $x \rightarrow 1-0$ . Then*

$$(8) \quad \liminf_{u \rightarrow \infty} \left\{ \sum_{n=0}^{\infty} a_n x^n - \sum_{n \leq u} a_n \right\} \geq p \liminf_{n \rightarrow \infty} a_n n \log \log n,$$

and the constant  $p$  is the best possible.

Since  $\sum_0^\infty a_n x^n$  is bounded above,

$$\begin{aligned} \sum_{n=0}^{\infty} a_n x^n - \sum_{n \leq u} a_n &= x^{-u} \left\{ \sum_{n=0}^{\infty} a_n x^n - x^u \sum_{n \leq u} a_n \right\} - (x^{-u} - 1) \sum_{n=0}^{\infty} a_n x^n \\ &\geq \exp\left(\frac{r}{(\log u)^p}\right) \left\{ \sum_{n=0}^{\infty} a_n x^n - x^u \sum_{n \leq u} a_n \right\} \\ &\quad + O\left(\frac{1}{(\log u)^p}\right), \end{aligned}$$

and (8) follows from Theorem 3 since  $\exp(r/(\log u)^p) \rightarrow 1$  as  $u \rightarrow \infty$ .

If we take  $a_n$  as in (7), we know that

$$\liminf_{u \rightarrow \infty} \left\{ \sum_{n=0}^{\infty} a_n x^n - x^u \sum_{n \leq u} a_n \right\} = p \liminf_{n \rightarrow \infty} a_n n \log \log n,$$

and

$$\begin{aligned} \left\{ \sum_{n=0}^{\infty} a_n x^n - x^u \sum_{n \leq u} a_n \right\} - \left\{ \sum_{n=0}^{\infty} a_n x^n - \sum_{n \leq u} a_n \right\} \\ = (1 - x^u) \sum_{n \leq u} \frac{1}{n \log \log n} \\ = O\left(\frac{1}{(\log u)^p}\right) O\left(\frac{\log u}{\log \log u}\right) \\ = O\left(\frac{1}{\log \log u}\right) = o(1) \end{aligned}$$

as  $u \rightarrow \infty$ . Hence

$$\liminf_{u \rightarrow \infty} \left\{ \sum_{n=0}^{\infty} a_n x^n - \sum_{n \leq u} a_n \right\} = p \liminf_{n \rightarrow \infty} a_n n \log \log n,$$

and since the additional condition imposed in this theorem is obviously satisfied, this completes the proof of the theorem.

We obtain the smallest constant in (3) if we take  $p=1$ . It is easy

to see that we cannot take  $p < 1$  here, for

$$\begin{aligned}
 i(u) &\cong \sum_{3 \leq n \leq u/\log \log u} \frac{x^n - x^u}{n \log \log n} \\
 &\cong \left\{ \exp\left(-\frac{r}{(\log u)^p \log \log u}\right) \right. \\
 &\quad \left. - \exp\left(-\frac{r}{(\log u)^p}\right) \right\} \sum_{3 \leq n \leq u/\log \log u} \frac{1}{n \log \log n} \\
 &= \left\{ \frac{r}{(\log u)^p} + o\left(\frac{1}{(\log u)^p}\right) \right\} \left\{ \frac{\log u}{\log \log u} + o\left(\frac{\log u}{\log \log u}\right) \right\} \\
 &= \frac{r}{(\log u)^{p-1} \log \log u} \{1 + o(1)\} \rightarrow \infty
 \end{aligned}$$

if  $p < 1$ . However, we can still obtain an inequality if we take

$$x = \exp(-q \log \log u / u \log u) \quad (q > 0).$$

**THEOREM 5.** *If*

$$x = \exp(-q \log \log u / u \log u),$$

where  $q$  is a fixed positive number, then

$$\liminf_{u \rightarrow \infty} \left\{ \sum_{n=0}^{\infty} a_n x^n - x^u \sum_{n \leq u} a_n \right\} \geq (q + 1) \liminf_{n \rightarrow \infty} a_n n \log \log n.$$

It is clear that we have to prove that

$$i(u) = \sum_{3 \leq n \leq u} \frac{x^n - x^u}{n \log \log n} + \sum_{n > u} \frac{x^n}{n \log \log n} \rightarrow q + 1$$

as  $u \rightarrow \infty$ . We have

$$\begin{aligned}
 i(u) &\leq (1 - x^u) \sum_{3 \leq n \leq u} \frac{1}{n \log \log n} + \frac{1}{\log \log u} \sum_{u < n \leq u \log u / \log \log u} \frac{1}{n} \\
 &\quad + \frac{1}{\log \log u} \sum_{n > -1/\log x} \frac{x^n}{n} \\
 &= \left\{ \frac{q \log \log u}{\log u} + o\left(\frac{\log \log u}{\log u}\right) \right\} \left\{ \frac{\log u}{\log \log u} + o\left(\frac{\log u}{\log \log u}\right) \right\} \\
 &\quad + \frac{\log \log u + o(\log \log u)}{\log \log u} + o(1) = q + 1 + o(1)
 \end{aligned}$$

as  $u \rightarrow \infty$ . On the other hand

$$\begin{aligned} t(u) &\geq \sum_{3 \leq n \leq u/\log \log u} \frac{x^n - x^u}{n \log \log n} + \sum_{u < n \leq u \log u / (\log \log u)^2} \frac{x^n}{n \log \log n} \\ &> \left\{ \exp\left(-\frac{q}{\log u}\right) - \exp\left(-\frac{q \log \log u}{\log u}\right) \right\} \\ &\quad \cdot \sum_{3 \leq n \leq u/\log \log u} \frac{1}{n \log \log n} \\ &\quad + \frac{\exp(-q/\log \log u)}{\log \log u + o(1)} \sum_{u < n \leq u \log u / (\log \log u)^2} \frac{1}{n} \\ &= q + 1 + o(1) \end{aligned}$$

as  $u \rightarrow \infty$ . This completes the proof.

It is not difficult to see that if  $x = e^{-v}$ , where

$$v = v(u) > 0, \quad v = o\left(\frac{\log \log u}{u \log u}\right),$$

$$1/v = O(u \log u),$$

then

$$\liminf_{u \rightarrow \infty} \left\{ \sum_{n=0}^{\infty} a_n x^n - x^u \sum_{n \leq u} a_n \right\} \geq \liminf_{n \rightarrow \infty} a_n n \log \log n,$$

and if  $\sum_0^{\infty} a_n x^n$  is bounded above as  $x \rightarrow 1-0$ ,

$$\liminf_{u \rightarrow \infty} \left\{ \sum_{n=0}^{\infty} a_n x^n - \sum_{n \leq u} a_n \right\} \geq \liminf_{n \rightarrow \infty} a_n n \log \log n.$$

This is the most precise inequality of this form that we can obtain, and the constant 1 corresponds to Agnew's constant  $\bar{p}(\log 2) = .96804 \dots$ .

In conclusion we show that the factor  $n \log \log n$  in the expression  $\liminf_{n \rightarrow \infty} a_n n \log \log n$  is the smallest that can occur in such one-sided inequalities, whatever function  $x = x(u)$  we choose.

Suppose that we try to obtain an inequality involving

$$\liminf_{n \rightarrow \infty} \frac{a_n n \log \log n}{\phi(n)}$$

where  $\phi(n) \rightarrow \infty$  as  $n \rightarrow \infty$ . We shall have to find, if possible, a function  $w(u) > 0$  such that if  $x = e^{-1/w}$ , then

$$s(u) \equiv \sum_{3 \leq n \leq u} \frac{\phi(n)(x^n - x^u)}{n \log \log n} + \sum_{n > u} \frac{\phi(n)x^n}{n \log \log n}$$

is bounded as  $u \rightarrow \infty$ . We show that this cannot be done.

Let  $n_0 (\geq 3)$  be such that  $\phi(n) \geq 1$  for  $n \geq n_0$ . Then

$$\begin{aligned} s(u) &\geq \sum_{n_0 \leq n \leq u/\log \log u} \frac{x^n - x^u}{n \log \log n} \\ &> \frac{1}{\log \log u} \left\{ \exp\left(-\frac{u}{w \log \log u}\right) \right. \\ &\quad \left. - \exp\left(-\frac{u}{w}\right) \right\} \sum_{n_0 \leq n \leq u/\log \log u} \frac{1}{n} \\ &= \frac{(u/w) \log u}{\log \log u} \{1 + o(1)\} \end{aligned}$$

as  $u \rightarrow \infty$ . Thus for  $s(u)$  to be bounded we must have

$$(9) \quad w > k \frac{u \log u}{\log \log u}$$

for some positive constant  $k$ , as  $u \rightarrow \infty$ . In particular  $w/u$  must tend to infinity with  $u$ , and so

$$\begin{aligned} s(u) &\geq \sum_{u < n \leq w} \frac{\phi(n)x^n}{n \log \log n} > \frac{e^{-1} \min_{u < v \leq w} \phi(v)}{\log \log w} \sum_{u < n \leq w} \frac{1}{n} \\ &= \frac{e^{-1} F(u) \log(w/u)}{\log \log w} \{1 + o(1)\} \end{aligned}$$

as  $u \rightarrow \infty$ , where  $F(u) = \min_{u < v \leq w(u)} \phi(v) \rightarrow \infty$  as  $u \rightarrow \infty$ . If  $s(u)$  is to be bounded as  $u \rightarrow \infty$ ,  $w(u)$  must certainly satisfy the inequality

$$(10) \quad \log \frac{w}{u} < K \frac{\log \log w}{F(u)}$$

for some constant  $K$ , when  $u$  is sufficiently large. We show that the inequalities (9) and (10) are incompatible.

From (10) it follows, in particular, that  $\log(w/u) < 2^{-1} \log w$ , that is to say,  $w < u^2$ , for large  $u$ . Hence

$$\log(w/u) < (K/F(u)) (\log \log u + \log 2) < (1/2) \log \log u$$

for sufficiently large  $u$  since  $F(u) \rightarrow \infty$  as  $u \rightarrow \infty$ . We thus have  $w < u(\log u)^{1/2}$  as  $u \rightarrow \infty$ , and this is clearly incompatible with (9).



## REFERENCES

1. R. P. Agnew, *Abel transforms of Tauberian series*, Duke Math. J. vol. 12 (1945) pp. 27–36.
2. ———, *Abel transforms and partial sums of Tauberian series*, Ann. of Math. (2) vol. 50 (1949) pp. 110–117.
3. H. Hadwiger, *Über ein Distanztheorem bei der  $A$ -Limitierung*, Comment. Math. Helv. vol. 16 (1943–1944) pp. 209–213.
4. ———, *Die Retardierungserscheinung bei Potenzreihen und Ermittlung zweier Konstanten Tauberscher Art*, Comment. Math. Helv. vol. 20 (1947) pp. 319–332.
5. ———, *Über eine Konstante Tauberscher Art*, Revista Matemática Hispano-Americana (4) vol. 7 (1947) pp. 65–69.
6. G. H. Hardy and J. E. Littlewood, *Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive*, Proc. London Math. Soc. (2) vol. 13 (1914) pp. 174–191.
7. P. Hartman, *Tauber's theorem and absolute constants*, Amer. J. Math. vol. 69 (1947) pp. 599–606.
8. J. E. Littlewood, *The converse of Abel's theorem on power series*, Proc. London Math. Soc. (2) vol. 9 (1910) pp. 434–448.
9. A. Tauber, *Ein Satz aus der Theorie der unendlichen Reihen*, Monatshefte für Mathematik und Physik vol. 8 (1897) pp. 273–277.
10. T. Vijayaraghavan, *Converse theorems on summability*, J. London Math. Soc. vol. 2 (1927) pp. 215–222.
11. A. Wintner, *On Tauber's theorem*, Comment. Math. Helv. vol. 20 (1947) pp. 216–222.

HARVARD UNIVERSITY