SOME INEQUALITIES RELATED TO ABEL'S METHOD OF SUMMATION

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1. It is well known that if

$$x = e^{-1/u}, \tag{1}$$

then there exists a constant ρ such that

$$\limsup_{u\to\infty} \left| \sum_{n=0}^{\infty} a_n x^n - \sum_{n\leqq u} a_n \right| \leqq \rho \limsup_{n\to\infty} |na_n| \tag{2}$$

for any series $\sum a_n$. This inequality is the source of Tauber's o-converse of Abel's theorem [Tauber **9**]. It is also the source of the following theorem of Vijayaraghavan [**10**, Theorem 1]:

THEOREM 1. *Suppose that the series* $\sum a_n x^n$ *is convergent for* $0<x<1$, *to the sum* $f(x)$ *say, and that, for some fixed real number* θ, $e^{i\theta}f(x)\to+\infty$ *as* $x\to1-0$. *Suppose further that* $a_n=O(1/n)$ *as* $n\to\infty$. *Then* $e^{i\theta}\sum_{n\leqq u}a_n\to+\infty$ *as* $u\to\infty$.

Theorem 1 may be stated rather less precisely as follows: *If the series* $\sum a_n$ *is summable* (A) *to the sum* s *with* $|s|=\infty$, *and if* $a_n=O(1/n)$, *then* $\sum a_n=s$. In this form it is seen to be an analogue, for infinite s, of Littlewood's well known O-Tauberian theorem for Abel summability [Littlewood **8**]. Vijayaraghavan showed that the corresponding analogue of the Hardy-Littlewood "one-sided" Tauberian theorem for Abel summability [Hardy and Littlewood **6**] is not true. He proved the following theorem [Vijayaraghavan **10**, Theorem 3], and showed by an example that it is "best possible."

THEOREM 2. *Suppose that the series* $\sum a_n x^n$ *is convergent for* $0<x<1$, *to the sum* $f(x)$ *say, that* $f(x)\to-\infty$ *as* $x\to1-0$, *and that the numbers* a_n *are real and satisfy the inequality*

$$a_n > -\frac{K}{n\log\log n}$$

when $n\geqq3$. *Then*

$$\sum_{n\leqq u} a_n\to-\infty \text{ as } u\to\infty.$$

It is the object of this paper to obtain an inequality related to

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Theorem 2 in the same way as the inequality (2) is related to Theorem 1.

Recently the inequality (2) and similar ones have received considerable attention [1; 2; 3; 4; 5; 7; 11]. Hartman [7] has found the best possible value $\bar{\rho}$ for the constant ρ in (2) ($\bar{\rho}=1.01598\cdots$). Agnew [1; 2; 2 contains an account of the previous work on the subject] has shown that if (1) is replaced by $x=e^{-q/u}$, then (2) remains true with ρ depending on q, and he has shown that the best possible value $\bar{\rho}(q)$ is least when $q=\log 2$ ($\bar{\rho}(\log 2)=0.96804\cdots$).

2. Theorem 2 is clearly a corollary of the theorem:

THEOREM 3. *Suppose that the series $\sum a_n x^n$ is convergent for $0<x<1$, and that the numbers a_n are real. Let*

$$x = \exp\left(-\frac{r}{u(\log u)^p}\right),$$

where $u>0$, and p and r are any fixed real numbers satisfying $p\geqq 1$ and $r>0$. Then

$$\liminf_{u\to\infty}\left\{\sum_{n=0}^{\infty} a_n x^n - x^u \sum_{n\leqq u} a_n\right\} \geqq p \liminf_{n\to\infty} a_n n \log\log n, \tag{3}$$

and the factor p on the right-hand side is the smallest for which the inequality is true.

The theorem is obviously true if the right-hand side is equal to $-\infty$. We may therefore suppose that

$$\liminf_{n\to\infty} a_n n \log\log n = \alpha > -\infty.$$

Let $\beta<\alpha$. Then for all but a finite number of the terms a_n we shall have

$$a_n n \log\log n > \beta. \tag{4}$$

It will be sufficient to show that the left side of (3) is greater than $p\beta$.

Since

$$x^u = \exp\left(-\frac{r}{(\log u)^p}\right) \to 1$$

as $u\to\infty$, we may change a finite number of the terms a_n without changing the value of the left side of (3), and we shall suppose that $a_0=a_1=a_2=0$, and that (4) holds for $n\geqq 3$.

Then

$$
\begin{aligned}
(5)\qquad \sum_{n=0}^{\infty} a_n x^n - x^u \sum_{n\leqq u} a_n &= \sum_{n\leqq u} a_n(x^n - x^u) + \sum_{n>u} a_n x^n \\
&> \beta\left\{\sum_{3\leqq n\leqq u} \frac{x^n - x^u}{n \log\log n} + \sum_{n>u} \frac{x^n}{n\log\log n}\right\} \\
&= \beta t(u)
\end{aligned}
$$

say. We have now to show that $t(u)\to p$ as $u\to\infty$.

Since $0<x<1$ and $u(\log u)^p = -r/\log x$,

$$
\begin{aligned}
(6)\qquad t(u) \leqq (1 - x^u) \sum_{3\leqq n\leqq u} \frac{1}{n\log\log n} &+ \frac{1}{\log\log u} \sum_{u\leqq n\leqq u(\log u)^p} \frac{1}{n} \\
&+ \frac{1}{\log\log u} \sum_{n>-r/\log x} \frac{x^n}{n}.
\end{aligned}
$$

Now

$$
1 - x^u = 1 - \exp\left(-\frac{r}{(\log u)^p}\right) = O\left(\frac{1}{(\log u)^p}\right),
$$

and

$$
\sum_{3\leqq n\leqq u} \frac{1}{n\log\log n} < 1 + \int_3^u \frac{dv}{v\log\log v} = O\left(\frac{\log u}{\log\log u}\right),
$$

so that the first term on the right side of (6) is

$$
O\left(\frac{1}{(\log u)^{p-1}\log\log u}\right) = O\left(\frac{1}{\log\log u}\right)
$$

since $p\geqq 1$. Since $r>0$,

$$
\begin{aligned}
\sum_{n>-r/\log x} \frac{x^n}{n} < 1 + \int_{-r/\log x}^{\infty} x^t \frac{dt}{t} &= 1 + \int_1^\infty x^{-vr/\log x} \frac{dv}{v} \\
&= 1 + \int_1^\infty e^{-rv} v^{-1} dv < \infty,
\end{aligned}
$$

and so the third term on the right side of (6) is $O(1/\log\log u)$. It remains to consider the second term. We have

$$
\sum_{1\leqq n\leqq y} \frac{1}{n} = \log y + O(1)
$$

as $y\to\infty$, and so

$$\sum_{u<n\leqq u(\log u)^p} \frac{1}{n} = \sum_{n\leqq u(\log u)^p} \frac{1}{n} - \sum_{n\leqq u} \frac{1}{n}$$
$$= \log \{u(\log u)^p\} - \log u + O(1)$$
$$= p \log \log u + O(1)$$

as $u \to \infty$. It follows that the second term is equal to

$$p + O\left(\frac{1}{\log \log u}\right)$$

as $u \to \infty$. Hence, by (6), we have

$$t(u) \leqq p + O\left(\frac{1}{\log \log u}\right) = p + o(1)$$

as $u \to \infty$. On the other hand

$$t(u) \geqq \sum_{u<n\leqq u(\log u)^p/\log \log u} \frac{x^n}{n \log \log n}$$
$$\geqq \frac{\exp(-1/\log \log u)}{\log \log \{u(\log u)^p/\log \log u\}} \sum_{u<n\leqq u(\log u)^p/\log \log u} \frac{1}{n}$$
$$= \frac{1 + O(1/\log \log u)}{\log \log u + O(\log \log u/\log u)} \{p \log \log u + O(\log \log \log u)\}$$
$$= p + O\left(\frac{\log \log \log u}{\log \log u}\right)$$
$$= p + o(1)$$

as $u \to \infty$. Thus

$$t(u) \to p \text{ as } u \to \infty,$$

and (3) is proved.

The example

$$a_n = \begin{cases} 0 & (n = 0, 1, 2), \\ -1/n \log \log n & (n \geqq 3) \end{cases} \tag{7}$$

proves that the inequality (5) is the best possible, and, consequently, that p is the best possible constant in (3).

This completes the proof of Theorem 3.

If we impose another condition on the series $\sum_0^\infty a_n$, we can remove the factor x^u from the left side of (3), and so obtain a closer analogy with (2).

THEOREM 4. *Suppose that the hypotheses of Theorem 3 are fulfilled, and that, in addition, the sum of the series* $\sum_0^\infty a_n x^n$ *is bounded above as* $x \to 1-0$. *Then*

$$\liminf_{u\to\infty} \left\{ \sum_{n=0}^{\infty} a_n x^n - \sum_{n\leqq u} a_n \right\} \geqq p \liminf_{n\to\infty} a_n n \log\log n, \tag{8}$$

and the constant p *is the best possible.*

Since $\sum_0^\infty a_n x^n$ is bounded above,

$$\begin{aligned}\sum_{n=0}^{\infty} a_n x^n - \sum_{n\leqq u} a_n &= x^{-u}\left\{ \sum_{n=0}^{\infty} a_n x^n - x^u \sum_{n\leqq u} a_n \right\} - (x^{-u}-1)\sum_{n=0}^{\infty} a_n x^n \\ &\geqq \exp\left(\frac{r}{(\log u)^p}\right)\left\{ \sum_{n=0}^{\infty} a_n x^n - x^u \sum_{n\leqq u} a_n \right\} \\ &\quad + O\left(\frac{1}{(\log u)^p}\right),\end{aligned}$$

and (8) follows from Theorem 3 since $\exp(r/(\log u)^p) \to 1$ as $u\to\infty$.

If we take a_n as in (7), we know that

$$\liminf_{u\to\infty} \left\{ \sum_{n=0}^{\infty} a_n x^n - x^u \sum_{n\leqq u} a_n \right\} = p \liminf_{n\to\infty} a_n n \log\log n,$$

and

$$\begin{aligned}\left\{ \sum_{n=0}^{\infty} a_n x^n - x^u \sum_{n\leqq u} a_n \right\} &- \left\{ \sum_{n=0}^{\infty} a_n x^n - \sum_{n\leqq u} a_n \right\} \\ &= (1-x^u)\sum_{n\leqq u} \frac{1}{n\log\log n} \\ &= O\left(\frac{1}{(\log u)^p}\right) O\left(\frac{\log u}{\log\log u}\right) \\ &= O\left(\frac{1}{\log\log u}\right) = o(1)\end{aligned}$$

as $u\to\infty$. Hence

$$\liminf_{u\to\infty} \left\{ \sum_{n=0}^{\infty} a_n x^n - \sum_{n\leqq u} a_n \right\} = p \liminf_{n\to\infty} a_n n \log\log n,$$

and since the additional condition imposed in this theorem is obviously satisfied, this completes the proof of the theorem.

We obtain the smallest constant in (3) if we take $p=1$. It is easy to see that we cannot take $p<1$ here, for

$$
\begin{aligned}
t(u) &\geqq \sum_{3\leqq n\leqq u/\log\log u} \frac{x^n - x^u}{n\log\log n} \\
&\geqq \left\{\exp\left(-\frac{r}{(\log u)^p \log\log u}\right)\right. \\
&\qquad \left.-\exp\left(-\frac{r}{(\log u)^p}\right)\right\} \sum_{3\leqq n\leqq u/\log\log u} \frac{1}{n\log\log n} \\
&= \left\{\frac{r}{(\log u)^p} + o\left(\frac{1}{(\log u)^p}\right)\right\}\left\{\frac{\log u}{\log\log u} + o\left(\frac{\log u}{\log\log u}\right)\right\} \\
&= \frac{r}{(\log u)^{p-1}\log\log u}\{1+o(1)\} \to \infty
\end{aligned}
$$

if $p<1$. However, we can still obtain an inequality if we take

$$
x = \exp(-q\log\log u/u\log u) \qquad (q>0).
$$

THEOREM 5. *If*

$$
x = \exp(-q\log\log u/u\log u),
$$

where q is a fixed positive number, then

$$
\liminf_{u\to\infty}\left\{\sum_{n=0}^{\infty} a_n x^n - x^u \sum_{n\leqq u} a_n\right\} \geqq (q+1)\liminf_{n\to\infty} a_n n\log\log n.
$$

It is clear that we have to prove that

$$
t(u) = \sum_{3\leqq n\leqq u} \frac{x^n - x^u}{n\log\log n} + \sum_{n>u}\frac{x^n}{n\log\log n} \to q+1
$$

as $u\to\infty$. We have

$$
\begin{aligned}
t(u) &\leqq (1-x^u)\sum_{3\leqq n\leqq u}\frac{1}{n\log\log n} + \frac{1}{\log\log u}\sum_{u<n\leqq u\log u/\log\log u}\frac{1}{n} \\
&\qquad + \frac{1}{\log\log u}\sum_{n>-1/\log x}\frac{x^n}{n} \\
&= \left\{\frac{q\log\log u}{\log u} + o\left(\frac{\log\log u}{\log u}\right)\right\}\left\{\frac{\log u}{\log\log u} + o\left(\frac{\log u}{\log\log u}\right)\right\} \\
&\qquad + \frac{\log\log u + o(\log\log u)}{\log\log u} + o(1) = q+1+o(1)
\end{aligned}
$$

as $u \to \infty$. On the other hand

$$t(u) \geqq \sum_{3 \leqq n \leqq u/\log\log u} \frac{x^n - x^u}{n \log \log n} + \sum_{u < n \leqq u \log u/(\log\log u)^2} \frac{x^n}{n \log\log n}$$
$$> \left\{\exp\left(-\frac{q}{\log u}\right) - \exp\left(-\frac{q \log\log u}{\log u}\right)\right\}$$
$$\cdot \sum_{3 \leqq n \leqq u/\log\log u} \frac{1}{n \log\log n}$$
$$+ \frac{\exp(-q/\log\log u)}{\log\log u + o(1)} \sum_{u < n \leqq u \log u/(\log\log u)^2} \frac{1}{n}$$
$$= q + 1 + o(1)$$

as $u \to \infty$. This completes the proof.

It is not difficult to see that if $x = e^{-v}$, where

$$v = v(u) > 0, \qquad v = o\left(\frac{\log\log u}{u \log u}\right),$$
$$1/v = O(u \log u),$$

then

$$\liminf_{u\to\infty} \left\{\sum_{n=0}^{\infty} a_n x^n - x^u \sum_{n \leqq u} a_n\right\} \geqq \liminf_{n\to\infty} a_n n \log\log n,$$

and if $\sum_0^\infty a_n x^n$ is bounded above as $x \to 1-0$,

$$\liminf_{u\to\infty} \left\{\sum_{n=0}^{\infty} a_n x^n - \sum_{n \leqq u} a_n\right\} \geqq \liminf_{n\to\infty} a_n n \log\log n.$$

This is the most precise inequality of this form that we can obtain, and the constant 1 corresponds to Agnew's constant $\bar{p}(\log 2) = .96804 \cdots$.

In conclusion we show that the factor $n \log\log n$ in the expression $\liminf_{n\to\infty} a_n n \log\log n$ is the smallest that can occur in such one-sided inequalities, whatever function $x = x(u)$ we choose.

Suppose that we try to obtain an inequality involving

$$\liminf_{n\to\infty} \frac{a_n n \log\log n}{\phi(n)}$$

where $\phi(n) \to \infty$ as $n \to \infty$. We shall have to find, if possible, a function $w(u) > 0$ such that if $x = e^{-1/w}$, then

$$s(u) \equiv \sum_{3 \leqq n \leqq u} \frac{\phi(n)(x^n - x^u)}{n \log \log n} + \sum_{n>u} \frac{\phi(n) x^n}{n \log \log n}$$

is bounded as $u \to \infty$. We show that this cannot be done.

Let $n_0(\geqq 3)$ be such that $\phi(n) \geqq 1$ for $n \geqq n_0$. Then

$$\begin{aligned} s(u) &\geqq \sum_{n_0 \leqq n \leqq u/\log \log u} \frac{x^n - x^u}{n \log \log n} \\ &> \frac{1}{\log \log u} \left\{ \exp\left(- \frac{u}{w \log \log u} \right) \right. \\ &\quad \left. - \exp\left(- \frac{u}{w} \right) \right\} \sum_{n_0 \leqq n \leqq u/\log \log u} \frac{1}{n} \\ &= \frac{(u/w) \log u}{\log \log u} \{1 + o(1)\} \end{aligned}$$

as $u \to \infty$. Thus for $s(u)$ to be bounded we must have

$$w > k \frac{u \log u}{\log \log u} \tag{9}$$

for some positive constant k, as $u \to \infty$. In particular w/u must tend to infinity with u, and so

$$\begin{aligned} s(u) &\geqq \sum_{u < n \leqq w} \frac{\phi(n) x^n}{n \log \log n} > \frac{e^{-1} \min_{u < \nu \leqq w} \phi(\nu)}{\log \log w} \sum_{u < n \leqq w} \frac{1}{n} \\ &= \frac{e^{-1} F(u) \log (w/u)}{\log \log w} \{1 + o(1)\} \end{aligned}$$

as $u \to \infty$, where $F(u) = \min_{u < \nu \leqq w(u)} \phi(\nu) \to \infty$ as $u \to \infty$. If $s(u)$ is to be bounded as $u \to \infty$, $w(u)$ must certainly satisfy the inequality

$$\log \frac{w}{u} < K \frac{\log \log w}{F(u)} \tag{10}$$

for some constant K, when u is sufficiently large. We show that the inequalities (9) and (10) are incompatible.

From (10) it follows, in particular, that $\log (w/u) < 2^{-1} \log w$, that is to say, $w < u^2$, for large u. Hence

$$\log (w/u) < (K/F(u)) (\log \log u + \log 2) < (1/2) \log \log u$$

for sufficiently large u since $F(u) \to \infty$ as $u \to \infty$. We thus have $w < u(\log u)^{1/2}$ as $u \to \infty$, and this is clearly incompatible with (9).

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