ON A CLASS OF INTEGRAL EQUATIONS

JOANNE ELLIOTT

1. Introduction. We shall discuss the integral equation

$$(1)\quad g(x) = P\cdot\frac{1}{\pi}\int_{-a}^{+a}\frac{f(t)}{t-x}\,dt + \lambda\int_{-a}^{+a} h(x, t)f(t)dt, \ -a < x < +a,$$

where g and h are given and f is unknown. The number a may be finite or infinite. The first integral is taken in the sense of a Cauchy principal value. For $h(x, t)\equiv 0$, this reduces to a well known integral equation of airfoil theory and has been treated extensively, mainly by Fourier methods, cf. [2]. E. Reissner [3] has applied this method to (1) in the case where $h(x, t)$ is an odd polynomial in $x-t$ and a is finite. He showed that a formal expansion of the assumed solution may be obtained in this case by solving finitely many linear equations. We may note that if $\lambda=1$ and $a=1$ and

$$h(x, t) = \frac{1}{2}\cot\pi\left(\frac{x-t}{2}\right) - \frac{\pi^{-1}}{x-t},$$

we obtain another well known equation.

It would seem of interest to consider the more general case.[1] The use of Fourier expansions suggests Hilbert space methods, and it is natural to modify the equation formally by introducing a suitable weight function $w(x)$. That is, instead of equation (1), we consider the equivalent equation

$$(2)\qquad g(x) = P\cdot\frac{1}{\pi}\int_{-a}^{+a}\frac{f(t)w(t)}{t-x}\,dt + \lambda\int_{-a}^{+a} h(x, t)f(t)w(t)dt.$$

Of all the choices of weight functions the most suitable turns out to be the one which makes

$$(3)\qquad Tf = P\cdot\frac{1}{\pi}\int_{-a}^{+a}\frac{f(t)w(t)}{t-x}\,dt$$

an isometric mapping from one Hilbert space onto another. If $a=\infty$, then $w(t)\equiv 1$ has this property. In the finite interval if we let $a=1$

Presented to the Society, December 28, 1951; received by the editors June 14, 1951.

[1] The author has recently found that Šerman [5] has considered a similar problem, but has used a different approach. He does not use Hilbert space, and his conditions are of a local character.

for simplicity, it will be shown that this is achieved by choosing

$$w(x) = (1-x)^{1/2}(1+x)^{-1/2}.$$[2]

The transformation T^{-1} will reduce (2) to a Fredholm equation. Two systems of Jacobi polynomials provide the necessary tools.

§§2 and 3 will be devoted to the case $a < \infty$ and the fourth to the case $a = \infty$.

2. **The Hilbert spaces for** $a < \infty$. We desire to find a weight function $w(x)$ such that (3) will transform a set of polynomials orthonormal with respect to $w(x)$ into a set of polynomials orthonormal with respect to another weight function $w_1(x)$. It is easily seen that (3) will transform a polynomial of degree n into another polynomial of degree n if and only if

$$P \cdot \frac{1}{\pi} \int_{-1}^{+1} \frac{w(t)}{t - x} dt = \text{const.} \neq 0. \tag{4}$$

Recall that we have put $a = 1$ here for simplicity. All admissible solutions of this equation are given by

$$w(t) = c_1(1 - t)^{1/2}(1 + t)^{-1/2} + c_2(1 - t^2)^{-1/2}$$

where c_1 and c_2 are constants. We shall choose

$$w(t) = (1 - t)^{1/2}(1 + t)^{-1/2}. \tag{5}$$

This will lead to

$$w_1(t) = (1 + t)^{1/2}(1 - t)^{-1/2} \tag{6}$$

as Lemma 1 will show. It will be clear from symmetry that the rôles of (5) and (6) could be interchanged. A system of polynomials orthonormal with respect to the weight function (5) are the Jacobi polynomials $P_n^{(1/2,-1/2)}(x)$ which we shall denote by $P_n(x)$. The polynomials corresponding to (6) are the Jacobi polynomials $P_n^{(-1/2,1/2)}(x)$ and will be denoted by $Q_n(x)$. These polynomials are defined by

$$\begin{aligned} P_n(\cos\theta) &= \alpha_n \frac{\sin\,[(n + 1/2)\theta]}{\sin\theta/2}, \\ Q_n(\cos\theta) &= \alpha_n \frac{\cos\,[(n + 1/2)\theta]}{\cos\theta/2} \end{aligned} \tag{7}$$

where α_n is a norming constant.

We define

[2] For general a, use $w(x) = (a-x)^{1/2}(a+x)^{-1/2}$.

(8) $$T_1 g = - P\cdot\frac{1}{\pi}\int_{-1}^{+1}\frac{g(t)w_1(t)}{t-x}\,dt.$$

The following two lemmas are contained in much more general results of Achieser [1]. In proving Lemma 2 we use Achieser's argument with the simplifications permitted by our special weight functions. His method for Lemma 1 is unnecessary in our case.

LEMMA 1. *With the weight functions defined in* (5) *and* (6), *we have*

(9) $$TP_n = -Q_n,$$

(10) $$-T_1 Q_n = P_n.$$

PROOF. To prove (9), we have to show that

(11) $$P\cdot\frac{1}{\pi}\int_0^{\pi}\frac{\sin\,[(n+1/2)\theta]\sin\theta/2}{\cos\theta-\cos\phi}\,d\theta = -\frac{1}{2}\,\frac{\cos\,[(n+1/2)\phi]}{\cos\phi/2}.$$

We may rewrite the left side as

(12) $$P\cdot\frac{1}{2\pi}\int_0^{\pi}\frac{\{\cos n\theta-\cos\,[(n+1)\theta]\}}{\cos\theta-\cos\phi}\,d\theta$$

and use the well known formula

(13) $$\frac{1}{\pi}P\cdot\int_0^{\pi}\frac{\cos m\theta}{\cos\theta-\cos\phi}\,d\theta = \frac{\sin m\phi}{\sin\phi},\qquad 0<\phi<\pi,$$

to obtain the result. We may prove (10) in a similar way.

LEMMA 2. *The transformation* (3) *maps the Hilbert space* $H[w]$ *of functions which are square integrable with respect to* $w(x)$ *isometrically onto the Hilbert space* $H[w^{-1}]$ *of functions which are square integrable with respect to* $1/w(x)$. *The inverse of* T *is* T_1.

PROOF. Any element $f\in H[w]$ can be expressed as $f(x) = \sum_{j=0}^{\infty} c_j P_j(x)$, the series converging in norm. We let $f_n(x) = \sum_{j=0}^{n} c_j P_j(x)$ and $g_n(x) = -\sum_{j=0}^{n} c_j Q_j(x)$. From Lemma 1, we know that $g_n(x) = Tf_n(x)$. We know also that g_n converges to an element $g\in H[w]$ as $n\to\infty$, that is, $g(x) = -\sum_{j=0}^{\infty} c_j Q_j(x)$. We cannot conclude, however, that $g = Tf$ until we know that T is a continuous transformation from $H[w]$ to $H[w_1]$. To show this, we first observe that

$$f(x)w(x)\in L_{2r/(r+1)}$$

for any $1<r<2$, by Hölder's inequality. From a well known theorem of Marcel Riesz, cf. [5, p. 132],

$$\int_{-\infty}^{+\infty} | g_n(x) - g_m(x) |^{2r/(r+1)} dx$$

$$\leqq A_r \int_{-1}^{+1} | [f_n(x) - f_m(x)] w(x) |^{2r/(r+1)} dx.$$

Hence g_n converges to Tf in the mean of $2r/(r+1)$. Since g_n converges to g in the norm of $H[w]$, it follows that $g = Tf$ almost everywhere.

3. Reduction to a Fredholm equation. Suppose that

$$\int_{-1}^{+1} | h(x, t) |^2 w_1(x) dx < \infty, \qquad -1 < t < +1,$$

$$\int_{-1}^{+1} \int_{-1}^{+1} | h(x, t) |^2 w_1(x) w(t) dx dt < \infty, \tag{14}$$

$$\int_{-1}^{+1} | g(x) |^2 w_1(x) dx < \infty.$$

In what follows all series expansions are to be considered as series converging in the norms of the corresponding Hilbert spaces. We can then expand h and g as follows:

$$h(x, t) = \sum_{i,j} c_{ij} Q_i(x) P_j(t), \quad g(x) = \sum_i g_i Q_i(x). \tag{15}$$

Suppose a solution $f(t) \in H[w]$ of equation (2) exists. Applying T_1 to both sides we have

$$\tilde{g}(x) = f(x) + \lambda \int_{-1}^{+1} \tilde{h}(x, t) f(t) w(t) dt \qquad \text{a.e.} \tag{16}$$

where $\tilde{h}(x, t) = -\sum_{i,j} c_{ij} P_i(x) P_j(t)$ and $\tilde{g} = T_1 g$. Conversely, we may apply T to both sides of (16), to conclude that the solutions must satisfy equation (2). Therefore *equation* (16) *is a Fredholm equation equivalent to* (2). Furthermore, if $c_{ij} = c_{ji}$, then the new equation will have a symmetric kernel.

We see also that (16) or (2) is equivalent to the system of linear equations

$$-g_k = f_k - \lambda \sum_j c_{kj} f_j, \qquad k = 0, 1, 2, \cdots. \tag{17}$$

Condition (14) on $h(x, t)$ implies that $\sum_{i,j} |c_{ij}^2| < \infty$. Hence the linear transformation on $l^{(2)}$ with matrix $[c_{ij}]$ is completely continuous. It is well known (cf. [**4**]) that in this case, for each fixed λ, the study of (17) may be reduced to the study of a finite system of equations. A condition which assures a unique solution of (2) is $\sum_{i,j} |c_{ij}^2| < |\lambda|^{-1}$. If $h(x, t)$ is a polynomial in $x-t$ with real coefficients α_i, then this reduces to

$$\sum_{i,j} \alpha_i\alpha_j \binom{i+j}{[(i+j)/2]} \binom{i+j+1}{[(i+j+2)/2]} \frac{(-1)^{i+j}\pi^2}{2^{i+j}} < |\lambda|^{-1}$$

where $[x]$ denotes the greatest integral part of x.

4. **The infinite interval.** In this case, we again reduce (1) to an equation of the Fredholm type. *We shall assume that*

$$\int_{-\infty}^{+\infty} |h(x, t)|^2 dx < \infty \tag{18}$$

for all t and is in $L[-\infty, +\infty]$.

Instead of the transformation (3), we shall use

$$Tf = P \cdot \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(t)}{t - x} dt, \tag{19}$$

the Hilbert transform (cf. [6, chap. V]). The inverse of T in $L_2[-\infty, +\infty]$ is given by

$$T_1 g = -P \cdot \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{g(t)}{t - x} dt, \tag{20}$$

and if $g = Tf$, then

$$\int_{-\infty}^{+\infty} |g(t)|^2 dt = \int_{-\infty}^{+\infty} |f(t)|^2 dt. \tag{21}$$

We shall use the following lemma.

LEMMA 2. *Let $f(t) \in L_2[-\infty, +\infty]$. If $F(t)$ is the Fourier transform of f, then the Fourier transform of Tf is given by $-iF(x)$ sgn x.*

A proof of this lemma may be found in [6, chap. V, Theorems 90 and 91].

We apply T_1 to both sides of (1). We first wish to show that

$$T_1\left\{\int_{-\infty}^{+\infty} h(x, t) f(t) dt\right\} = -\int_{-\infty}^{+\infty} f(t) \left\{\text{P.V. } \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{h(u, t)}{u - x} du\right\} dt. \tag{22}$$

Note that condition (18) assures that the function in braces on the left side of (22) is in L_2.

Let

$$h(x, t) = \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{(2\pi)^{1/2}} \int_{-A}^{+A} e^{-ixu} H(u, t) du. \tag{23}$$

Then, assuming for the moment that the following formal operations are justified, we have

$$\begin{aligned} \int_{-\infty}^{+\infty} f(t) h(x, t) dt &= \int_{-\infty}^{+\infty} f(t) \left\{ \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{(2\pi)^{1/2}} \int_{-A}^{+A} e^{-ixu} H(u, t) du \right\} dt \\ &= \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{(2\pi)^{1/2}} \int_{-A}^{+A} e^{-ixu} \left\{ \int_{-\infty}^{+\infty} f(t) H(u, t) dt \right\} du. \end{aligned} \tag{24}$$

The function in braces in the last integral is in L_2 by Parseval's formula. By Lemma 2 this gives

$$\begin{aligned} T_1 \left\{ \int_{-\infty}^{+\infty} f(t) h(x, t) dt \right\} &= \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{(2\pi)^{1/2}} \int_{-A}^{+A} e^{-ixu} i \operatorname{sgn} u \left\{ \int_{-\infty}^{+\infty} f(t) H(u, t) dt \right\} du \\ &= i \int_{-\infty}^{+\infty} f(t) \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{(2\pi)^{1/2}} \int_{-A}^{+A} H(u, t) e^{-ixu} \operatorname{sgn} u du \Big\} dt, \end{aligned} \tag{25}$$

which is just (22).

We must now prove (24). We have

$$\begin{aligned} &\int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} f(t) \left[h(x, t) - \frac{1}{(2\pi)^{1/2}} \int_{-A}^{+A} e^{-ixu} H(u, t) du \right] dt \right|^2 dx \\ &\leqq \|f\|^2 \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} \left| h(x, t) - \frac{1}{(2\pi)^{1/2}} \int_{-A}^{+A} e^{-ixu} H(u, t) du \right|^2 dt \right\} dx, \end{aligned} \tag{26}$$

which is equal to

$$\|f\|^2 \int_{-\infty}^{+\infty} \left\{ \int_{|u|>A} | H(u, t) |^2 du \right\} dt. \tag{27}$$

For the proof of this last statement, see [6, p. 75]. That (27) goes to zero with A follows from (18). This completes the proof of (24).

The equation then reduces to

$$\tilde{g}(x) = f(x) + \lambda \int_{-\infty}^{+\infty} f(t)\tilde{h}(x, t)dt \qquad \text{a.e.} \tag{28}$$

where $\tilde{g} = T_1 g$ and $\tilde{h}(x, t) = T_1 h(x, t)$ where the last transformation is taken with respect to x. Furthermore $\tilde{h}(x, t)$ also satisfies (18). By a similar argument on (28) with T it is clear that (28) is equivalent to (1) with $a = \infty$. The integral transform in (28) is completely continuous and the equation (28) will have a unique solution in L_2 provided

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\tilde{h}(x, t)|^2 dxdt < |\lambda|^{-1}.$$

5. **Comparison with other methods.** The usual treatment of (3) and Reissner's treatment of (1) is equivalent to using instead of (5) and (6) the weight functions

$$w(x) = (1 - x^2)^{-1/2} \quad \text{and} \quad w_1(x) = (1 - x^2)^{1/2}$$

respectively, and reducing to a system of linear equations by direct substitution into (2). We cannot use the mapping T as we did before since with this weight function, $w(x)$, any constant is taken into zero by T. This transformation takes the Tchebycheff polynomials $T_n(x)$ of the first kind into the Tchebycheff polynomials $U_{n-1}(x)$ of the second kind. The system of equations obtained by this procedure is no longer of the Fredholm type (17).

References

1. N. I. Achieser, *On some inversion formulas for singular integrals* (in Russian), Bull. Acad. Sci. URSS (Izvestia) Sér. Math. vol. 9 (1945).

2. G. Hamel, *Integralgleichungen*, Springer, 1949.

3. E. Reissner, *Solution of a class of singular integral equations*, Bull. Amer. Math. Soc. vol. 51 (1945).

4. F. Riesz, *Les systèmes d'équations linéaires à une infinité d'inconnues*, Gauthier-Villars, 1913.

5. D. I. Šerman, *On a case of regularization of singular integral equations* (in Russian), Akademiya Nauk SSSR. Prikladnaya Matematika i Mehanika vol. 15 (1951) pp. 75–82.

6. E. C. Titchmarsh, *Introduction to the theory of the Fourier integral*, Oxford, 1948.

Mount Holyoke College and
Swarthmore College