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TWO NOTES ON NILPOTENT GROUPS

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I

We extend a theorem of Rédei and Szép.[1] Our proof is quite straightforward, and employs a method of considerably more general applicability.[2]

The *lower central series* of a group G is formed by taking $G_1 = G$, and successively defining G_{n+1} to be the commutator (G_n, G). G is *nilpotent* if some $G_{N+1} = 1$. If A and B are subgroups of G, $A \vee B$ is the subgroup generated by the elements of A and of B together, and A^m the subgroup generated by the mth powers of elements of A.

THEOREM. *Let A and K be subgroups of a nilpotent group G, and let $A^{m^e} = 1$ for some integer m^e. Then, for any $n \geq 1$,*

$$(A \vee K)_n = (A^m \vee K)_n \textit{ implies } (A \vee K)_n = K_n.$$

We may clearly suppose that $G = A \vee K$. The elements of G_r can be written as products of commutators of *order r*:

$$(x_1, \cdots, x_r) = ((\cdots ((x_1, x_2), x_3) \cdots, x_{r-1}), x_r).$$

Let C_r be the subgroup generated by those commutators for which

Received by the editors January 7, 1952.

[1] L. Rédei and J. Szép, Monatshefte für Mathematik vol. 55, p. 200. The present proof avoids "counting arguments" and the attendant finiteness conditions; for $n = 1$ the present argument reduces substantially to that of Rédei and Szép. We remark that the hypothesis $A^{m^e} = 1$ admits various modifications.

[2] The basic idea of "expanding" words in commutators of ascending order has been exploited by P. Hall, Proc. London Math. Soc. vol. 36, p. 29; and by O. Grün, J. Reine Angew. Math. vol. 182, p. 158. See also W. Magnus, Monatschefte für Mathematik vol. 47, p. 307, and K. T. Chen, Proceedings of the American Mathematical Society vol. 3, p. 44.

some x_i is in A, and D_r by those for which some x_i is in A^m. From the identity

$$(x, yz) = (x, y)(x, z)(z, x, y)$$

it follows that all commutators are linear in the x_i, modulo commutators of higher order. In particular, it follows that

$$(A \vee K)_n = K_n \vee C_n,$$

$$(A^m \vee K)_n = K_n \vee D_n,$$

$$D_n \subset C_n^m \vee C_{n+1}, \tag{1}$$

and, since $A^{m^e} = 1$, that

$$C_n^{m^e} \subset C_{n+1}. \tag{2}$$

From the hypothesis that $(A \vee K)_n = (A^m \vee K)_n$, hence that $K_n \vee C_n = K_n \vee D_n$, we have $C_n \subset K_n \vee D_n$ and, from (1),

$$C_n \subset K_n \vee C_n^m \vee C_{n+1}. \tag{3}$$

By the evident rule $(L \vee M)^m \subset L^m \vee M^m \vee (L, M)$, from

$$C_n \subset K_n \vee C_n^r \vee C_{n+1}$$

we deduce that

$$C_n^m \subset K_n^m \vee C_n^{rm} \vee C_{n+1}^m \vee G_{2n},$$

$$C_n^m \subset K_n \vee C_n^{rm} \vee C_{n+1},$$

and, by (3), that

$$C_n \subset K_n \vee C_n^{rm} \vee C_{n+1}.$$

Applying this argument $e-1$ times to (3) gives

$$C_n \subset K_n \vee C_n^{m^e} \vee C_{n+1},$$

whence, by (2),

$$C_n \subset K_n \vee C_{n+1}. \tag{4}$$

From the Lie-Jacobi congruences

$$(x, y)(y, x) = 1, \qquad (x, y, z)(y, z, x)(z, x, y) \equiv 1 \pmod{G_4},$$

it follows that every $(x_1, \cdots, x_{k+1})$ with x_{k+1} in A is expressible, modulo G_{k+2}, as a product of such factors with x_i in A for some $i \leqq k$: in short,

$$C_{k+1} \subset (C_k, G) \vee C_{k+2} = (C_k, K) \vee (C_k, A) \vee C_{k+2}.$$

Assuming now

$$C_k \subset K_k \vee C_{k+1}$$

and substituting, we obtain

$$(C_k, K) \subset K_{k+1} \vee C_{k+2},$$
$$(C_k, A) \subset (K_k, A) \vee C_{k+2} \subset (C_k, K) \vee C_{k+2} \subset (C_k, K),$$

whence

$$C_{k+1} \subset K_{k+1} \vee C_{k+2}.$$

By iteration, it follows from (4) that

$$C_n \subset K_n \vee K_{n+1} \vee \cdots \vee K_N \vee C_{N+1} \subset K_n \vee C_{N+1}.$$

Since $G_{N+1} = 1$ by hypothesis,

$$C_n \subset K_n,$$

whence $K_n \vee C_n = K_n$ and $(A \vee K)_n = K_n$, as required.

II

By a uniform method[3] we establish easily two results that are fairly obvious from well known considerations, and a further result (Theorem 2.1) which answers for nilpotent groups a question regarding identical relations in groups that was raised by B. H. Neumann.[4]

We employ standard notation for commutators: $(x_1, \cdots, x_n) = (\cdots((x_1, x_2), x_3), \cdots, x_n)$, and for the lower central series: $G = G_1$, $G_{n+1} = (G_n, G)$.

LEMMA 1. *Let F be a finitely generated free group, and R a normal subgroup of F. Then, for each $n \geqq 1$, $R = [S_n, R_{n+1}]$, the normal subgroup generated by a finite set S_n together with $R_{n+1} = R \cap F_{n+1}$.*

PROOF. Proceed inductively from the vacuous case $n = 0$. Since F_{n+1}/F_{n+2} is a finitely generated abelian group, so is its subgroup R_{n+1}/R_{n+2}. Let $T = \{r_i\}$ be a finite set of elements of R_{n+1} such that the cosets $r_i R_{n+2}$ generate R_{n+1}/R_{n+2}. Evidently $R = [S_n, R_{n+1}]$ implies $R = [S_n, T, R_{n+2}]$.

THEOREM 1.1. *Every finitely generated nilpotent group is definable by a finite set of relations.*

[3] For the method, see references given in footnote 2.

[4] B. H. Neumann, Math. Ann. vol. 114, p. 506. Theorem 2.1 was announced by the author in Bull. Amer. Math. Soc. Abstract 57-4-278.

PROOF. If $G=F/R$ is nilpotent, say $G_{N+1}=1$, we have $R_{N+1} = F_{N+1} = [(x_1, \cdots, x_{N+1})$, all sets $x_1, \cdots, x_N$ of generators for $F]$. Hence $R=[S_N, R_{N+1}]$ is defined by a finite set of relations.

THEOREM 1.2. *In a finitely generated group which is known to be nilpotent*[5] *the word-problem is decidable.*

PROOF. Let $G=F/R$ and $G_{N+1}=1$. Suppose we have an expression for the word w in the form $w=r_{n-1}w_n$, where r_{n-1} is in R and w_n is in F_n. By reference to the finitely generated abelian group F_n/F_{n+1}, we can obtain an expression $w_n=r_nw_{n+1}$, r in R, w_{n+1} in F_{n+1}, if any such exists. Proceeding thus, either $w=r_1r_2 \cdots r_Nw_{N+1}$ in R, or else, for some n, $w=r_1r_2 \cdots r_{n-1}w_n$ where w_n is not in $[R, F_{n+1}]$ and hence w is not in R.

A normal subgroup W of the free group F is a *word group* if it is defined by certain words $w(\xi_1, \cdots, \xi_n)$ under all substitutions of elements of F for the ξ_i. For any group G, let F be a denumerably generated free group; the group W_G of *identical relations* for G is the normal subgroup of F defined by all words $w(\xi_1, \cdots, \xi_n)$ that equal 1 under all substitutions of elements of G for the ξ_i.

LEMMA 2. *Let F be a free group and W a word subgroup of F. Then, for each $n \geqq 1$, $W=\{S_n, W_{n+1}\}$, the word group defined by a finite set S_n of words, in at most n indeterminates, together with $W_{n+1}=W \cap F_{n+1}$.*

PROOF. Induction as for Lemma 1. Consider the set of all relations of the form

$$r = \prod_i c_i \cdot s \tag{1}$$

where the c_i are commutators of generators of F of order $n+1$, $\prod c_i \neq 1$, and s is in F_{n+2}. Each c_i contains at most $n+1$ generators. Let X be the set of generators occurring in some c_{i_0} in r. Substituting $x_k \to 1$ for all generators x_i not in X, we derive from r a relation

$$r' = \prod{}' c_i \cdot s' \tag{2}$$

where $\prod' c_i$ is a partial product of that occurring in (1) and contains

[5] It is understood that G is defined by a finite set of relations, whence a finite set for F_n modulo F_{n+1} can be obtained, say, by a simplification of the Reidemeister-Schreier process. It suffices for Theorem 2.1, in fact, to assume that the G_n have intersection 1. To see this, test, for $n=1, 2, \cdots$, the two conditions: (i) w is in $[R, F_n]$; and (ii) w is not equal, in F, to any product $\prod u_ir_iu_i^{-1}$ where the u_i and r_i together are of total length less than n. For some finite n either (i) must fail and so w is not in R, or (ii) must fail, whence w is in R.

at least the factor c_{i_0}. Therefore, in

$$r'' = r \cdot r'^{-1} = \prod{}'' c_i \cdot s'' \tag{3}$$

the product contains fewer factors than that in (1). If we repeat this construction, each relation (1) is obtained as a consequence of relations (2), each involving at most $n+1$ generators. Now, all the relations (2) are equivalent, for the purpose of defining W, to the corresponding relations (2″) in the generators $x_1, \cdots, x_{n+1}$, and, by Lemma 1, these possess a finite basis T modulo W_{n+2}. Thus, if $W = \{S_n, W_{n+1}\}$, then $W = \{S_n, T, W_{n+2}\}$.

THEOREM 2.1. *A nilpotent group G possesses a finite basis of identical relations.*

PROOF. If $G_{N+1}=1$, then $F_{N+1} \subset W_G \subset F$. By Lemma 2, $W = \{S_N, W_{N+1}\}$ where S_N is finite. But $W_{N+1} = F_{N+1}$ is defined by the single word $(\xi_1, \cdots, \xi_{N+1})$, whence W has a finite basis. We note that, by multiplying together all the words in this basis, taken with distinct indeterminates, we obtain a single word which constitutes a basis for W.

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