# A GENERALIZATION OF THE RIEMANN INTEGRAL

CASPER GOFFMAN

By considering certain kinds of sets as exceptional in the definition of the Riemann integral, a variety of possible generalizations of the integral concept for bounded functions is obtained, one of which is the Lebesgue integral. Such an extension has been discussed by E. H. Hanson [1], whose class of exceptional sets, however, does not include the Lebesgue integral as a member of the associated class of integrals.

We consider a property P of sets relative to intervals, which will always be closed on the left and open on the right. We write $(S, I)^{\mathrm{P}}$ or $(S, I)^{\overline{\mathrm{P}}}$ according as the set $S$ has the property P or does not have the property P relative to the interval $I$, and suppose that P is such that

(a) if $(S, I)^{\mathrm{P}}$ and $T \subset S$, then $(T, I)^{\mathrm{P}}$,

(b) if $(S, I)^{\mathrm{P}}$, then $(CS, I)^{\overline{\mathrm{P}}}$, where $CS$ is the complement of $S$.

We associate two interval functions with every admissible property P and bounded real function $f(x)$, defined on the interval $[0, 1)$, as follows:

$$\phi(\mathrm{P}, f; I) = \inf \left[ y \mid (E(f(x) > y), I)^{\mathrm{P}} \right]$$

and

$$\psi(\mathrm{P}, f; I) = \sup \left[ y \mid (E(f(x) < y), I)^{\mathrm{P}} \right],$$

where $E(f(x)>y)$ and $E(f(x)<y)$ are the sets of points for which $f(x)>y$ and $f(x)<y$, respectively.

We define the upper and lower P-integrals of $f(x)$ as the upper Burkill integral [2] of $\phi(\mathrm{P}, f; I)$ and the lower Burkill integral of $\psi(\mathrm{P}, f; I)$, respectively; i.e.,

$$\mathrm{P}\int^{*} f(x)dx = \mathrm{B}\int^{*} \phi(\mathrm{P}, f; I)$$

and

$$\mathrm{P}\int_{*} f(x)dx = \mathrm{B}\int_{*} \psi(\mathrm{P}, f; I).$$

If the upper and lower P-integrals of $f(x)$ are equal, we say that the P-integral, $\mathrm{P}\int f(x)dx$, exists, and we have

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$$\mathrm{P}\int f(x)dx = \mathrm{P}\int^{*} f(x)dx = \mathrm{P}\int_{*} f(x)dx.$$

We note in passing that the P-integral of $f(x)$ exists if and only if the Burkill integrals of $\phi(\mathrm{P}, f; I)$ and $\psi(\mathrm{P}, f; I)$ both exist and are equal.

For the case where $(S, I)^{\mathrm{P}}$ if the set $S\cap I$ is empty, the P-integral is the Riemann integral and, as we shall show, for the case where $(S, I)^{\mathrm{P}}$ if the relative exterior measure of $S$ in $I$ is less than $1/2$, the P-integral is the Lebesgue integral for bounded functions. Hanson considered those properties P for which $(S, I)^{\mathrm{P}}$ if $S\cap I$ belongs to a class $\mathcal{E}$ of sets such that

($\alpha$) if $E_1\in\mathcal{E}$ and $E_2\subset E_1$, then $E_2\in\mathcal{E}$,

($\beta$) if $E_n\in\mathcal{E}$, $n=1, 2, \cdots$, and $E=\bigcup_{n=1}^{\infty}E_n$, then $E\in\mathcal{E}$,

($\gamma$) if $E$ is an interval, then $E\notin\mathcal{E}$.

It is clear that every such property obeys our conditions (a) and (b). To see that no P-integral is the Lebesgue integral if it is obtained from a class $\mathcal{E}$ of sets satisfying the above conditions ($\alpha$), ($\beta$), and ($\gamma$), we first note that in order to yield the Lebesgue integral the class $\mathcal{E}$ must contain only sets of measure zero. For, if $\mathcal{E}$ contains a set $S$ whose exterior measure is positive, then the characteristic function of $S$ has Lebesgue integral different from zero and P-integral equal to zero. Now, suppose $\mathcal{E}$ contains only sets of measure zero. Let $S$ be any measurable set such that the measures of the intersections of $S$ and $CS$ with $I$ are both positive for every interval $I\subset[0, 1)$. Then the P-integral associated with the class $\mathcal{E}$ of sets does not exist for the characteristic function of $S$, but the Lebesgue integral of this function does exist.

There are properties P, which satisfy our conditions, for which the P-integral is not additive. For example, let $(S, I)^{\mathrm{P}}$ if $S\cap I$ is a proper subset of the rational numbers in $I$. Let $R$ be the set of rational numbers in $[0, 1)$ and let $Q$ be the set of those numbers in $R$ which are of the form $k/2^n$, where $k$ and $n$ are positive integers. Then, let

$$f(x) = \begin{cases} 1 & \text{if} \quad x\in Q, \\ 0 & \text{if} \quad x\notin Q \end{cases}$$

and

$$g(x) = \begin{cases} 1 & \text{if} \quad x\in R-Q, \\ 0 & \text{if} \quad x\notin R-Q. \end{cases}$$

Then

$$f(x) + g(x) = \begin{cases} 1 & \text{if} \quad x \in R, \\ 0 & \text{if} \quad x \notin R. \end{cases}$$

We may readily verify that $\mathrm{P}\int f(x)dx = 0$, $\mathrm{P}\int g(x)dx = 0$, and $\mathrm{P}\int^*(f(x) + g(x))dx = 1$.

However, the P-integral is additive if P is such that $(S, I)^{\mathrm{P}}$ and $(T, I)^{\mathrm{P}}$ imply $(S \cup T, I)^{\mathrm{P}}$. Indeed, we have the following theorem.

THEOREM 1. *If* P *has an associated* P′ *such that* $(S, I)^{\mathrm{P}'}$ *and* $(T, I)^{\mathrm{P}'}$ *imply* $(S \cup T, I)^{\mathrm{P}}$, *and the existence of the* P-*integral of a bounded function implies the existence of its* P′-*integral, then the existence of the* P-*integrals of* $f(x)$ *and* $g(x)$ *implies the existence of the* P-*integral of* $f(x)+g(x)$ *and the equality*

$$\mathrm{P}\int (f(x) + g(x))dx = \mathrm{P}\int f(x)dx + \mathrm{P}\int g(x)dx.$$

PROOF. It is an almost immediate consequence of the hypothesis that the P′-integral of a bounded function $f(x)$ exists and is equal to the P-integral of $f(x)$ if and only if the P-integral of $f(x)$ exists. Now, let $\epsilon > 0$. There is an $\eta > 0$ such that for every partition of $[0, 1)$ into intervals $I_1, I_2, \cdots, I_n$ whose lengths are less than $\eta$, there are real numbers $y_1, y_2, \cdots, y_n; z_1, z_2, \cdots, z_n$ such that

$$(E(f(x) > y_k), I_k)^{\mathrm{P}'} \quad \text{and} \quad (E(g(x) > z_k), I_k)^{\mathrm{P}'}$$

for every $k = 1, 2, \cdots, n$, and

$$\sum_{k=1}^{n} y_k l(I_k) < \mathrm{P}'\int f(x)dx + \frac{\epsilon}{2}, \qquad \sum_{k=1}^{n} z_k l(I_k) < \mathrm{P}'\int g(x)dx + \frac{\epsilon}{2},$$

where $l(I_k)$ is the length of $I_k$. Now,

$$E(f(x) + g(x) > y_k + z_k) \subset E(f(x) > y_k) \cup E(g(x) > z_k)$$

so that

$$(E(f(x) + g(x) > y_k + z_k), I_k)^{\mathrm{P}}.$$

Hence,

$$\sum_{k=1}^{n} \phi(\mathrm{P}, f + g; I_k) \leqq \sum_{k=1}^{n} (y_k + z_k)l(I_k) = \sum_{k=1}^{n} y_k l(I_k) + \sum_{k=1}^{n} z_k l(I_k)$$

$$< \mathrm{P}\int f(x)dx + \mathrm{P}\int g(x)dx + \epsilon,$$

since $\mathrm{P}\int f(x)dx = \mathrm{P}'\int f(x)dx$ and $\mathrm{P}\int g(x)dx = \mathrm{P}'\int g(x)dx$. As this holds for every partition of $[0, 1)$ into intervals whose lengths are less than $\eta$, it follows that

$$\mathrm{P}\int^{*}(f(x) + g(x))dx < \mathrm{P}\int f(x)dx + \mathrm{P}\int g(x)dx + \epsilon.$$

Similarly,

$$\mathrm{P}\int_{*}(f(x) + g(x))dx > \mathrm{P}\int f(x)dx + \mathrm{P}\int g(x)dx - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $\mathrm{P}\int(f(x)+g(x))dx$ exists, and

$$\mathrm{P}\int (f(x) + g(x))dx = \mathrm{P}\int f(x)dx + \mathrm{P}\int g(x)dx.$$

COROLLARY. *The* P*-integral is additive if* P *is such that* $(S, I)^{\mathrm{P}}$ *and* $(T, I)^{\mathrm{P}}$ *imply* $(S \cup T, I)^{\mathrm{P}}$.

PROOF. The conditions of Theorem 1 are satisfied with $\mathrm{P} = \mathrm{P}'$.

We now show that the examples of properties P given above yield the Riemann and Lebesgue integrals for bounded functions.

THEOREM 2. *If* $(S, I)^{\mathrm{P}}$ *means that* $S \cap I$ *is empty, then the* P*-integral of* $f(x)$ *agrees with its Riemann integral.*

The proof is clear.

THEOREM 3. *If* $(S, I)^{\mathrm{P}}$ *means that* $S$ *has relative exterior measure less than* $1/2$ *in* $I$, *then the* P*-integral of a bounded function* $f(x)$ *exists if and only if its Lebesgue integral exists, and then the two integrals are equal.*

PROOF. By a theorem of Kamke [3], $f(x)$ is measurable, hence integrable, since it is bounded, if and only if it is approximately continuous almost everywhere.

Suppose $f(x)$ is approximately continuous almost everywhere. Then, for every $\epsilon > 0$, for every $x$, except for those belonging to a certain set of measure zero, for every interval $I$ containing $x$ which is sufficiently small, the set of points $y$ for which $f(y) > f(x) + \epsilon$ and $f(y) < f(x) - \epsilon$ are both of relative exterior measure less than $1/2$ in the interval $I$. A routine argument now shows that the P-integral of $f(x)$ exists and is equal to its Lebesgue integral.

If $f(x)$ is nonmeasurable, then since the set of points of approximate continuity of any function is measurable, the set of points of approximate discontinuity of $f(x)$ is measurable and of positive measure. It follows that there are real numbers $r$ and $s$, with $r>s$, such that the sets $E(f(x)>r)$ and $E(f(x)<s)$ both have positive upper exterior metric density at all the points of a set of positive measure. By the Lebesgue Density Theorem, these sets both have exterior metric density which exists and is equal to one at all the points of a set of measure $k>0$. It readily follows that

$$\mathrm{P}\int^{*} f(x)dx - \mathrm{P}\int_{*} f(x)dx > k(r-s) > 0,$$

so that the P-integral of $f(x)$ does not exist.

The same proof shows that for the property P′, where $(S, I)^{\mathrm{P}'}$ means that the relative exterior measure of $S$ is less than $1/4$ in $I$, the P′-integral is also the Lebesgue integral. Hence, the property P of Theorem 3 satisfies the conditions of Theorem 1.

Our purpose here was to introduce the P-integral and show that there is a property P for which the P-integral is the Lebesgue integral. We hope to give a detailed discussion of the partial order of abstract P-integrals elsewhere.

## References

1. E. H. Hanson, *The r-limit*, Ph.D. Dissertation, The Ohio State University, 1934.

2. S. Saks, *Theory of the integral*, trans. by L. C. Young, Warsaw, 1937, p. 165.

3. E. Kamke, *Zur Definition der approximativ stetigen Funktionen*, Fund. Math. vol. 10 (1927).

University of Oklahoma