We shall say that an analytic function \( f(z) \) has property \( J \) at a point \( z_1 \) if the sequence of derivatives \( \{ f^{(n)}(z_1) \} \), \( n = 0, 1, \cdots \), takes on only a finite number of distinct values. An entire function of the form

\[
f(z) = Q(z) + \sum_{j=0}^{m-1} A_j \exp \{ \omega^j z \},
\]

where \( Q(z) \) is a polynomial and \( \omega = \exp \{ 2\pi i/m \} \), has property \( J \) at every point, but a function having property \( J \) at one point is not necessarily a special exponential sum (as we shall term a function of form (1)). We give three theorems whose conditions relate property \( J \) to special exponential sums.

**Theorem 1.** If \( f(z) \) has property \( J \) at two points, then it is a special exponential sum.

**Theorem 2.** If \( f(z) \) has property \( J \) at a point \( z_1 \), and if at a second point \( z_2 \neq z_1 \) infinitely many derivatives are equal, then \( f(z) \) is a special exponential sum.

**Theorem 3.** Let \( r, \sigma, A \) be arbitrary positive numbers, with \( r \) an integer and \( \sigma \leq 1 \). Then there is an integer \( V = V[r, \sigma, A] \) with the following property: Let \( f(z) \) have property \( J \) at \( z_1 \), where the distinct values of \( \{ f^{(n)}(z_1) \} \) are \( a_1, \cdots, a_t \), with \( t \leq r \) and

\[
\min |a_i - a_j| \geq \sigma \max |a_i - a_j| \quad (i \neq j = 1, 2, \cdots, t).
\]

If for a value \( z_2 \) in \( 0 < |z_1 - z_2| \leq A \) there are more than \( V \) equal quantities in the sequence \( \{ f^{(n)}(z_2) \} \), then \( f(z) \) is a special exponential sum.

It is clear that Theorem 3 implies 2 and 2 implies 1, so it suffices to establish 3. Since a translation in the independent variable does not alter the essential conditions, we may suppose that \( z_1 = 0 \) and (changing the letter) that \( z_2 = a \). Then

\[
f(z) = \sum_{n=0}^{\infty} c_n \frac{z^n}{n!} = \sum_{n=0}^{\infty} d_n \frac{(z - a)^n}{n!},
\]

where \( c_n = f^{(n)}(0) \) and \( d_n = f^{(n)}(a) \). If we differentiate (3) \( n \) times and set \( z = a \), we obtain the linear relations

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Define
\[ \delta_k(A) = \sum_{p=1}^{\infty} \frac{A^p}{(k + 1)(k + 2) \cdots (k + p)}. \]
Since \( \delta_k(A) \to 0 \) as \( k \to \infty \), there is a smallest value \( k = K = K[a, A] \)
such that
\[ \delta_K(A) < \sigma. \]
We shall show that a possible choice of \( V \) is \( V = r^K \). Set
\[ P_{n,s}(a) = c_n + \frac{a}{1!} c_{n+1} + \cdots + \frac{a^s}{s!} c_{n+s}. \]
Then
\[ d_n - d_q = \{ P_{n,K-1}(a) - P_{q,K-1}(a) \} \]
\[ + \frac{a^K}{K!} \left( (c_{n+K} - c_{q+K}) + \sum_{s=K+1}^{\infty} (c_{n+s} - c_{q+s}) \frac{a^{s-K}}{(K + 1) \cdots s} \right). \]
Since \( |c_{n+s} - c_{q+s}| \leq \max |a_i - a_j| \), and \( |a| \leq A \), the last sum in (8) cannot exceed in magnitude the quantity \( \delta_K(A) \cdot \max |a_i - a_j| \).
Now suppose \( d_{n_1} = d_{n_2} = \cdots = d_{n_v} \), where \( v > r^K \). Since each \( c_j \) has one of the values \( a_1, \cdots, a_t \), therefore for fixed \( s \) there are at most \( r^{s+1} \) different expressions \( P_{n,s}(a) \). Hence of the expressions \( P_{n_j,K-1}(a), \)
\( j=1, \cdots, v \), at least two are equal; say for \( n = n_a, n_b \). Taking \( n = n_a, q = n_b \) in (8), we see that
\[ c_{n_a+K} - c_{n_b+K} = \sum_{s=K+1}^{\infty} (c_{n_a+s} - c_{n_b+s}) \frac{a^{s-K}}{(K + 1) \cdots s}; \]
so
\[ |c_{n_a+K} - c_{n_b+K}| \leq \delta_K(A) \cdot \max |a_i - a_j| < \sigma \cdot \max |a_i - a_j| \]
This is in contradiction to (2) unless \( c_{n_a+K} = c_{n_b+K} \). We may therefore rewrite (9) as
\[ (c_{n_a+K+1} - c_{n_b+K+1}) = -\sum_{s=K+2}^{\infty} (c_{n_a+s} - c_{n_b+s}) \frac{a^{s-K-1}}{(K + 2) \cdots s}, \]
and from this conclude that \( c_{n_a+K+1} = c_{n_b+K+1} \), and so on, with the result that
\[ c_{n+a+K+j} = c_{n+b+K+j}, \quad j = 0, 1, 2, \ldots. \]

Thus, beginning at least with the index \( n = n_a + K \), the sequence \( \{c_n\} \) is periodic; and it is an easy consequence that \( f(z) \) is a special exponential sum.

**Remarks.** (i) A theorem of Szegö states that if the coefficients \( \{c_n\} \) of the series \( F(z) = \sum_0^\infty c_n z^n \) take on only a finite number of distinct values, then either (a) \( F(z) \) has the circle \( |z| = 1 \) as cut, or (b) \( F(z) \) is a rational function of the form \( F(z) = P(z)/(1-z^m) \) where \( m \) is a positive integer and \( P(z) \) is a polynomial. Both cases arise, and this suggests the problem of assigning a further condition to insure (let us say) that case (b) holds. If we introduce the entire function \( f(z) = \sum_0^\infty c_n z^n/n! \) associated with \( F(z) \), then \( f(z) \) has property \( J \) at \( z = 0 \); and case (b) is easily seen to be equivalent to the condition that \( f(z) \) be a special exponential sum. Thus, the conditions of any one of Theorems 1, 2, 3 suffice to guarantee case (b).

(ii) Theorem 1 shows that relative to property \( J \) there are only three possibilities for an analytic function \( f(z) \): either it has property \( J \) at no point whatever, or at just one point, or at all points. Also, as noted by the referee, Theorem 1 can be formulated in this way: If an entire function of exponential type has for its indicator diagram a circle, center at the origin, then it cannot have property \( J \) at two points.

(iii) In the course of the proof of Theorem 3 it was shown that a permissible choice of \( V \) is \( V = V[r, \sigma, A] = r^K \), where \( K = K[\sigma, A] \). It would be of interest to determine, for given \( r, \sigma, A \), the smallest possible \( V \). It is conceivable that this minimum \( V \) is independent of one or more of the quantities \( r, \sigma, A \).

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