ON CERTAIN ENTIRE FUNCTIONS

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We shall say that an analytic function $f(z)$ has *property $\mathcal{J}$ at a point* z_1 if the sequence of derivatives $\{f^{(n)}(z_1)\}$, $n=0, 1, \cdots$, takes on only a finite number of distinct values. An entire function of the form

$$f(z) = Q(z) + \sum_{j=0}^{m-1} A_j \exp \{\omega^j z\}, \tag{1}$$

where $Q(z)$ is a polynomial and $\omega = \exp\{2\pi i/m\}$, has property $\mathcal{J}$ at every point, but a function having property $\mathcal{J}$ at one point is not necessarily a *special exponential sum* (as we shall term a function of form (1)). We give three theorems whose conditions relate property $\mathcal{J}$ to special exponential sums.

THEOREM 1. *If $f(z)$ has property $\mathcal{J}$ at two points, then it is a special exponential sum.*

THEOREM 2. *If $f(z)$ has property $\mathcal{J}$ at a point z_1, and if at a second point $z_2 \neq z_1$ infinitely many derivatives are equal, then $f(z)$ is a special exponential sum.*

THEOREM 3. *Let r, σ, A be arbitrary positive numbers, with r an integer and $\sigma \leqq 1$. Then there is an integer $V = V[r, \sigma, A]$ with the following property: Let $f(z)$ have property $\mathcal{J}$ at z_1 where the distinct values of $\{f^{(n)}(z_1)\}$ are $a_1, \cdots, a_t$, with $t \leqq r$ and*

$$\min |a_i - a_j| \geqq \sigma \cdot \max |a_i - a_j| \qquad (i \neq j = 1, 2, \cdots, t). \tag{2}$$

If for a value z_2 in $0 < |z_1 - z_2| \leqq A$ there are more than V equal quantities in the sequence $\{f^{(n)}(z_2)\}$, then $f(z)$ is a special exponential sum.

It is clear that Theorem 3 implies 2 and 2 implies 1, so it suffices to establish 3. Since a translation in the independent variable does not alter the essential conditions, we may suppose that $z_1 = 0$ and (changing the letter) that $z_2 = a$. Then

$$f(z) = \sum_0^\infty c_n \frac{z^n}{n!} = \sum_0^\infty d_n \frac{(z-a)^n}{n!}, \tag{3}$$

where $c_n = f^{(n)}(0)$ and $d_n = f^{(n)}(a)$. If we differentiate (3) n times and set $z = a$, we obtain the linear relations

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$$c_n + \frac{a}{1!} c_{n+1} + \frac{a^2}{2!} c_{n+2} + \cdots = d_n \qquad (n = 0, 1, 2, \cdots). \tag{4}$$

Define

$$\delta_k(A) = \sum_{p=1}^{\infty} \frac{A^p}{(k+1)(k+2)\cdots(k+p)}. \tag{5}$$

Since $\delta_k(A) \to 0$ as $k \to \infty$, there is a smallest value $k = K = K[\sigma, A]$ such that

$$\delta_K(A) < \sigma. \tag{6}$$

We shall show that a possible choice of V is $V = r^K$. Set

$$P_{n,s}(a) = c_n + \frac{a}{1!} c_{n+1} + \cdots + \frac{a^s}{s!} c_{n+s}. \tag{7}$$

Then

$$\begin{aligned} d_n - d_q &= \{P_{n,K-1}(a) - P_{q,K-1}(a)\} \\ &\quad + \frac{a^K}{K!}\left\{(c_{n+K} - c_{q+K}) + \sum_{s=K+1}^{\infty} (c_{n+s} - c_{q+s}) \frac{a^{s-K}}{(K+1)\cdots s}\right\}. \end{aligned} \tag{8}$$

Since $|c_{n+s} - c_{q+s}| \leqq \max |a_i - a_j|$, and $|a| \leqq A$, the last sum in (8) cannot exceed in magnitude the quantity $\delta_K(A) \cdot \max |a_i - a_j|$.

Now suppose $d_{n_1} = d_{n_2} = \cdots = d_{n_v}$, where $v > r^K$. Since each c_j has one of the values $a_1, \cdots, a_t$, therefore for fixed s there are at most r^{s+1} different expressions $P_{n,s}(a)$. Hence of the expressions $P_{n_j,K-1}(a)$, $j = 1, \cdots, v$, at least two are equal; say for $n = n_\alpha, n_\beta$. Taking $n = n_\alpha$, $q = n_\beta$ in (8), we see that

$$c_{n_\alpha+K} - c_{n_\beta+K} = -\sum_{s=K+1}^{\infty} (c_{n_\alpha+s} - c_{n_\beta+s}) \frac{a^{s-K}}{(K+1)\cdots s}; \tag{9}$$

so

$$|c_{n_\alpha+K} - c_{n_\beta+K}| \leqq \delta_K(A) \cdot \max |a_i - a_j| < \sigma \cdot \max |a_i - a_j|.$$

This is in contradiction to (2) unless $c_{n_\alpha+K} = c_{n_\beta+K}$. We may therefore rewrite (9) as

$$(c_{n_\alpha+K+1} - c_{n_\beta+K+1}) = -\sum_{s=K+2}^{\infty} (c_{n_\alpha+s} - c_{n_\beta+s}) \frac{a^{s-K-1}}{(K+2)\cdots s},$$

and from this conclude that $c_{n_\alpha+K+1} = c_{n_\beta+K+1}$, and so on, with the result that

$$c_{n_\alpha+K+j} = c_{n_\beta+K+j}, \qquad j = 0, 1, 2, \cdots. \tag{10}$$

Thus, beginning at least with the index $n = n_\alpha + K$, the sequence $\{c_n\}$ is periodic; and it is an easy consequence that $f(z)$ is a special exponential sum.

REMARKS. (i) A theorem[1] of Szegö states that *if the coefficients* $\{c_n\}$ *of the series* $F(z) = \sum_0^\infty c_n z^n$ *take on only a finite number of distinct values, then either* (a) $F(z)$ *has the circle* $|z| = 1$ *as cut, or* (b) $F(z)$ *is a rational function of the form* $F(z) = P(z)/(1 - z^m)$ *where* m *is a positive integer and* $P(z)$ *is a polynomial.* Both cases arise, and this suggests the problem of assigning a further condition to insure (let us say) that case (b) holds. If we introduce the entire function $f(z) = \sum_0^\infty c_n z^n/n!$ associated with $F(z)$, then $f(z)$ has property $\mathcal{J}$ at $z = 0$; and case (b) is easily seen to be equivalent to the condition that $f(z)$ be a special exponential sum. Thus, the conditions of any one of Theorems 1, 2, 3 suffice to guarantee case (b).

(ii) Theorem 1 shows that relative to property $\mathcal{J}$ there are only three possibilities for an analytic function $f(z)$: either it has property $\mathcal{J}$ at no point whatever, or at just one point, or at all points. Also, as noted by the referee, Theorem 1 can be formulated in this way: If an entire function of exponential type has for its indicator diagram a circle, center at the origin, then it cannot have property $\mathcal{J}$ at two points.

(iii) In the course of the proof of Theorem 3 it was shown that a permissible choice of V is $V = V[r, \sigma, A] = r^K$, where $K = K[\sigma, A]$. It would be of interest to determine, for given r, σ, A, the smallest possible V. It is conceivable that this minimum V is independent of one or more of the quantities r, σ, A.

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[1] G. Szegö, *Über Potenzreihen mit endlich vielen verschiedenen Koeffizienten*, Berl. Ber. (1922) pp. 88–91. A proof is also found in P. Dienes, *The Taylor series*, Oxford, 1931, pp. 324–327.