# ON A CONJECTURE CONCERNING DOUBLY STOCHASTIC MATRICES

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A doubly stochastic (d.s.) matrix is a real $n\times n$ matrix $P=(p_{ij})$ such that

$$(1)\qquad p_{ij}\geqq 0,\qquad 1\leqq i\leqq n,\ 1\leqq j\leqq n,$$

$$(2)\qquad \sum_j p_{ij}=1,\qquad 1\leqq i\leqq n,$$

and

$$(3)\qquad \sum_i p_{ij}=1,\qquad 1\leqq j\leqq n.$$

We introduce a partial order among d.s. matrices by defining

$$(4)\qquad P^1\prec P^3$$

to mean there exists a d.s. matrix $P^2$ such that

$$(5)\qquad P^1=P^2P^3.$$

We introduce a partial order among real vectors $a=(a_1,\cdots,a_n)$ of our real $n$-dimensional space $E$ by defining

$$(6)\qquad a\prec b$$

to mean for each real convex $\phi$

$$(7)\qquad \sum_i \phi(a_i)\leqq\sum_i\phi(b_i).$$

By [HLP, p. 89], $a\prec b$ if and only if there exists a d.s. matrix $P$ such that

$$a=Pb=\left(\sum_j p_{1j}b_j,\cdots,\sum_j p_{nj}b_j\right).$$

This implies that for each real $n$-vector $a$,

$$(8)\qquad P^1\prec P^3\rightarrow P^1a\prec P^3a.$$

Kakutani has raised the following conjecture.

CONJECTURE. If, for each real $n$-vector $a$, $P^1a\prec P^3a$, then $P^1\prec P^3$. By [HLP, p. 89] if the hypothesis is satisfied there exists a d.s.

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matrix $P_a^2$ such that

$$P^1 a = P_a^2 P^3 a. \tag{9}$$

The issue is to show that if there exists such a $P_a^2$ for each vector $a$, there exists a d.s. $P^2$ independent of $a$ such that $P^1 = P^2 P^3$.

Let $\mathcal{P}$ be the collection of vectors with non-negative components. Requirement (1) is equivalent to the requirement

$$P\mathcal{P} \subset \mathcal{P}. \tag{10}$$

If $e$ is the vector all of whose components are unity, then requirement (2) is equivalent to requirement

$$Pe = e, \tag{11}$$

i.e., $e$ is a characteristic vector of characteristic value unity. If $\bar{e}$ is the element of $\overline{E}$, the conjugate space of $E$, whose value at $a \in E$ is given by $(\bar{e}, a) = \sum_i a_i$, then $\bar{e}^\perp = \{a \mid (\bar{e}, a) = 0\}$ is the set of vectors $a \in E$ whose components add up to zero. Requirement (3) is equivalent to

$$P(\bar{e}^\perp) \subset \bar{e}^\perp. \tag{12}$$

Proof of conjecture. Suppose now $P^1 a \prec P^3 a$ for each $a \in E$. Consider a mapping $\psi$: $P^3 E \to P^1 E$ defined as follows: if $b \in P^3 E$, for some $a \in E$ we have $b = P^3 a$, then let $\psi(b) = P^1 a$. We first prove (i) that we have a valid definition, i.e., $\psi(b)$ is uniquely defined by the above and then we prove (ii) that $\psi$: $P^3 E \to P^1 E$ is a linear transformation.

In order to prove (i) suppose that $P^3 a' = P^3 a'' = b$. We wish to show that $P^1 a' = P^1 a''$. If $P^3 a' = P^3 a''$, then $P^3(a' - a'') = 0$, the zero vector. Since $P^1(a' - a'') \prec P^3(a' - a'')$, by (9) we deduce that $P^1(a' - a'') = 0$ and so $P^1 a' = P^1 a''$. Thus we have shown that $\psi$ is uniquely defined. The linearity (ii) of $\psi$ is now trivial. If $\alpha$ is a real scalar and $P^3 a = b$, then $P^3 \alpha a = \alpha b$ and $\psi(\alpha b) = P^1 \alpha a = \alpha P^1 a = \alpha \psi(b)$. Also if $P^3 a' = b'$ and $P^3 a'' = b''$, then $P^3(a' + a'') = b' + b''$ and $\psi(b' + b'') = P^1(a' + a'') = P^1 a' + P^1 a'' = \psi(b') + \psi(b'')$.

Suppose $P^3 a \in \mathcal{P}$. Since $P^1 a \prec P^3 a$ by (9) there exists a d.s. $P_a$ such that $P^1 a = P_a P^3 a \in \mathcal{P}$. Thus $\psi(\mathcal{P} \cap P^3 E) \subset \mathcal{P}$. Since $P^1$ and $P^3$ are d.s. matrices, $P^1 e = P^3 e = e$ and so $\psi(e) = e$. If by $(Pa)_i$ we denote the $i$th component of $Pa$, then by [HLP, p. 89] and the assumption $P^1 a \prec P^3 a$ we have

$$\sum_i (P^1 a)_i = \sum_i (P^3 a)_i.$$

In particular

$$\psi(\bar{e}^{\perp} \cap P^3E) \subset \bar{e}^{\perp}.$$

We can now extend $\psi$ to a function $\Psi$ on all of $E$ by letting $\Psi(b) = 0$ for each $b \in E \cap (P^3E)'$, the complement of $P^3E$, and $\Psi(b) = \psi(b)$ for each $b \in P^3E$. Now $\Psi$ is a linear transformation satisfying the requirements

$$\Psi(\mathcal{P}) \subset \mathcal{P}, \qquad \Psi(e) = e, \qquad \Psi(\bar{e}^{\perp}) \subset \bar{e}^{\perp}.$$

Therefore we can represent $\Psi$ by a d.s. matrix $P^2$. We now have $P^1a = P^2P^3a$ for each $a \in E$ and therefore $P^1 = P^2P^3$, thus establishing the conjecture.

It can readily be shown that if $P^1 \prec P^3$, then for each $j = 1, 2, \cdots, n$

$$(\overset{1}{p}_{1j}, \overset{1}{p}_{2j}, \cdots, \overset{1}{p}_{nj}) \prec (\overset{3}{p}_{1j}, \cdots, \overset{3}{p}_{nj}).$$

It would be interesting to establish the converse.

## Reference

G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, 1934.

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