

## ON A CONJECTURE CONCERNING DOUBLY STOCHASTIC MATRICES

S. SHERMAN

A doubly stochastic (d.s.) matrix is a real  $n \times n$  matrix  $P = (p_{ij})$  such that

$$(1) \quad p_{ij} \geq 0, \quad 1 \leq i \leq n, 1 \leq j \leq n,$$

$$(2) \quad \sum_j p_{ij} = 1, \quad 1 \leq i \leq n,$$

and

$$(3) \quad \sum_i p_{ij} = 1, \quad 1 \leq j \leq n.$$

We introduce a partial order among d.s. matrices by defining

$$(4) \quad P^1 < P^3$$

to mean there exists a d.s. matrix  $P^2$  such that

$$(5) \quad P^1 = P^2 P^3.$$

We introduce a partial order among real vectors  $a = (a_1, \dots, a_n)$  of our real  $n$ -dimensional space  $E$  by defining

$$(6) \quad a < b$$

to mean for each real convex  $\phi$

$$(7) \quad \sum_i \phi(a_i) \leq \sum_i \phi(b_i).$$

By [HLP, p. 89],  $a < b$  if and only if there exists a d.s. matrix  $P$  such that

$$a = Pb = \left( \sum_j p_{1j} b_j, \dots, \sum_j p_{nj} b_j \right).$$

This implies that for each real  $n$ -vector  $a$ ,

$$(8) \quad P^1 < P^3 \rightarrow P^1 a < P^3 a.$$

Kakutani has raised the following conjecture.

**CONJECTURE.** If, for each real  $n$ -vector  $a$ ,  $P^1 a < P^3 a$ , then  $P^1 < P^3$ .  
By [HLP, p. 89] if the hypothesis is satisfied there exists a d.s.

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matrix  $P_a^2$  such that

$$(9) \quad P^1 a = P_a^2 P^3 a.$$

The issue is to show that if there exists such a  $P_a^2$  for each vector  $a$ , there exists a d.s.  $P^2$  independent of  $a$  such that  $P^1 = P^2 P^3$ .

Let  $\mathcal{P}$  be the collection of vectors with non-negative components. Requirement (1) is equivalent to the requirement

$$(10) \quad P\mathcal{P} \subset \mathcal{P}.$$

If  $e$  is the vector all of whose components are unity, then requirement (2) is equivalent to requirement

$$(11) \quad Pe = e,$$

i.e.,  $e$  is a characteristic vector of characteristic value unity. If  $\bar{e}$  is the element of  $\bar{E}$ , the conjugate space of  $E$ , whose value at  $a \in E$  is given by  $(\bar{e}, a) = \sum_i a_i$ , then  $\bar{e}^\perp = \{a \mid (\bar{e}, a) = 0\}$  is the set of vectors  $a \in E$  whose components add up to zero. Requirement (3) is equivalent to

$$(12) \quad P(\bar{e}^\perp) \subset \bar{e}^\perp.$$

PROOF OF CONJECTURE. Suppose now  $P^1 a < P^3 a$  for each  $a \in E$ . Consider a mapping  $\psi: P^3 E \rightarrow P^1 E$  defined as follows: if  $b \in P^3 E$ , for some  $a \in E$  we have  $b = P^3 a$ , then let  $\psi(b) = P^1 a$ . We first prove (i) that we have a valid definition, i.e.,  $\psi(b)$  is uniquely defined by the above and then we prove (ii) that  $\psi: P^3 E \rightarrow P^1 E$  is a linear transformation.

In order to prove (i) suppose that  $P^3 a' = P^3 a'' = b$ . We wish to show that  $P^1 a' = P^1 a''$ . If  $P^3 a' = P^3 a''$ , then  $P^3(a' - a'') = 0$ , the zero vector. Since  $P^1(a' - a'') < P^3(a' - a'')$ , by (9) we deduce that  $P^1(a' - a'') = 0$  and so  $P^1 a' = P^1 a''$ . Thus we have shown that  $\psi$  is uniquely defined. The linearity (ii) of  $\psi$  is now trivial. If  $\alpha$  is a real scalar and  $P^3 a = b$ , then  $P^3 \alpha a = \alpha b$  and  $\psi(\alpha b) = P^1 \alpha a = \alpha P^1 a = \alpha \psi(b)$ . Also if  $P^3 a' = b'$  and  $P^3 a'' = b''$ , then  $P^3(a' + a'') = b' + b''$  and  $\psi(b' + b'') = P^1(a' + a'') = P^1 a' + P^1 a'' = \psi(b') + \psi(b'')$ .

Suppose  $P^3 a \in \mathcal{P}$ . Since  $P^1 a < P^3 a$  by (9) there exists a d.s.  $P_a$  such that  $P^1 a = P_a P^3 a \in \mathcal{P}$ . Thus  $\psi(\mathcal{P} \cap P^3 E) \subset \mathcal{P}$ . Since  $P^1$  and  $P^3$  are d.s. matrices,  $P^1 e = P^3 e = e$  and so  $\psi(e) = e$ . If by  $(Pa)_i$  we denote the  $i$ th component of  $Pa$ , then by [HLP, p. 89] and the assumption  $P^1 a < P^3 a$  we have

$$\sum_i (P^1 a)_i = \sum_i (P^3 a)_i.$$

In particular

$$\psi(\bar{e}^\perp \cap P^3 E) \subset \bar{e}^\perp.$$

We can now extend  $\psi$  to a function  $\Psi$  on all of  $E$  by letting  $\Psi(b) = 0$  for each  $b \in E \cap (P^3 E)'$ , the complement of  $P^3 E$ , and  $\Psi(b) = \psi(b)$  for each  $b \in P^3 E$ . Now  $\Psi$  is a linear transformation satisfying the requirements

$$\Psi(\mathcal{P}) \subset \mathcal{P}, \quad \Psi(e) = e, \quad \Psi(\bar{e}^\perp) \subset \bar{e}^\perp.$$

Therefore we can represent  $\Psi$  by a d.s. matrix  $P^2$ . We now have  $P^1 a = P^2 P^3 a$  for each  $a \in E$  and therefore  $P^1 = P^2 P^3$ , thus establishing the conjecture.

It can readily be shown that if  $P^1 < P^3$ , then for each  $j = 1, 2, \dots, n$

$$(\overset{1}{p}_{1j}, \overset{1}{p}_{2j}, \dots, \overset{1}{p}_{nj}) < (\overset{3}{p}_{1j}, \dots, \overset{3}{p}_{nj}).$$

It would be interesting to establish the converse.

#### REFERENCE

G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, 1934.

NAVY RESEARCH GROUP, LOCKHEED AIRCRAFT CORPORATION