ON A CONJECTURE CONCERNING DOUBLY STOCHASTIC MATRICES

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A doubly stochastic (d.s.) matrix is a real \( n \times n \) matrix \( P = (p_{ij}) \) such that

1. \( p_{ij} \geq 0 \), \( 1 \leq i \leq n, 1 \leq j \leq n \),
2. \( \sum_i p_{ij} = 1 \), \( 1 \leq i \leq n \),

and

3. \( \sum_j p_{ij} = 1 \), \( 1 \leq j \leq n \).

We introduce a partial order among d.s. matrices by defining

4. \( P_1 < P_2 \)

to mean there exists a d.s. matrix \( P_2 \) such that

5. \( P_1 = P_2 P_1 \).

We introduce a partial order among real vectors \( a = (a_1, \ldots, a_n) \) of our real \( n \)-dimensional space \( E \) by defining

6. \( a < b \)

to mean for each real convex \( \phi \)

7. \( \sum_i \phi(a_i) \leq \sum_i \phi(b_i) \).

By [HLP, p. 89], \( a < b \) if and only if there exists a d.s. matrix \( P \) such that

\[
a = Pb = \left( \sum_i p_{1i}b_i, \ldots, \sum_i p_{ni}b_i \right).
\]

This implies that for each real \( n \)-vector \( a \),

8. \( P_1 < P_2 \Rightarrow P_1 a < P_2 a \).

Kakutani has raised the following conjecture.

**Conjecture.** If, for each real \( n \)-vector \( a \), \( P_1 a < P_2 a \), then \( P_1 < P_2 \).

By [HLP, p. 89] if the hypothesis is satisfied there exists a d.s.

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matrix $P_a^2$ such that
\[ P^1a = P_a^2P^3a. \]

The issue is to show that if there exists such a $P_a^2$ for each vector $a$, there exists a d.s. $P^2$ independent of $a$ such that $P^1 = P^2P^3$.

Let $\mathcal{P}$ be the collection of vectors with non-negative components. Requirement (1) is equivalent to the requirement
\[ PP \subset \mathcal{P}. \]

If $e$ is the vector all of whose components are unity, then requirement (2) is equivalent to requirement
\[ Pe = e, \]

i.e., $e$ is a characteristic vector of characteristic value unity. If $\hat{e}$ is the element of $E$, the conjugate space of $E$, whose value at $a \in E$ is given by $(\hat{e}, a) = \sum a_i$, then $e^\perp = \{ a \mid (\hat{e}, a) = 0 \}$ is the set of vectors $a \in E$ whose components add up to zero. Requirement (3) is equivalent to
\[ P(e^\perp) \subset e^\perp. \]

Proof of conjecture. Suppose now $P^1a < P^3a$ for each $a \in E$. Consider a mapping $\psi: P^3E \rightarrow P^1E$ defined as follows: if $b \in P^3E$, for some $a \in E$ we have $b = P^3a$, then let $\psi(b) = P^1a$. We first prove (i) that we have a valid definition, i.e., $\psi(b)$ is uniquely defined by the above and then we prove (ii) that $\psi: P^3E \rightarrow P^1E$ is a linear transformation.

In order to prove (i) suppose that $P^3a' = P^3a'' = b$. We wish to show that $P^1a' = P^1a''$. If $P^3a' = P^3a''$, then $P^3(a' - a'') = 0$, the zero vector. Since $P^1(a' - a'') = P^3(a' - a'')$, by (9) we deduce that $P^1(a' - a'') = 0$ and so $P^1a' = P^1a''$. Thus we have shown that $\psi$ is uniquely defined. The linearity (ii) of $\psi$ is now trivial. If $\alpha$ is a real scalar and $P^3a = b$, then $P^3\alpha a = \alpha b$ and $\psi(\alpha b) = P^3\alpha a = \alpha P^1a = \alpha \psi(b)$. Also if $P^3a' = b'$ and $P^3a'' = b''$, then $P^3(a' + a'') = b' + b''$ and $\psi(b' + b'') = P^1(a' + a'') = P^1a' + P^1a'' = \psi(b') + \psi(b'')$.

Suppose $P^3a \in \mathcal{P}$. Since $P^1a < P^3a$ by (9) there exists a d.s. $P_a$ such that $P^1a = P_aP^3a \in \mathcal{P}$. Thus $\psi(P_a \cap P^1E) \subset \mathcal{P}$. Since $P^1$ and $P^3$ are d.s. matrices, $P^1e = P^3e = e$ and so $\psi(e) = e$. If by $(P_a)$, we denote the $i$th component of $P_a$, then by [HLP, p. 89] and the assumption $P^1a < P^3a$ we have
\[ \sum_i (P^1a)_i = \sum_i (P^3a)_i. \]
In particular

\[ \psi(\bar{e}_1 \cap P^3 E) \subset \bar{e}_1. \]

We can now extend \[ \psi \] to a function \[ \Psi \] on all of \[ E \] by letting \[ \Psi(b) = 0 \] for each \[ b \in E \cap (P^3 E)' \], the complement of \[ P^3 E \], and \[ \Psi(b) = \psi(b) \] for each \[ b \in P^3 E \]. Now \[ \Psi \] is a linear transformation satisfying the requirements

\[ \Psi(\mathcal{P}) \subset \mathcal{P}, \quad \Psi(e) = e, \quad \Psi(\bar{e}_1) \subset \bar{e}_1. \]

Therefore we can represent \[ \Psi \] by a d.s. matrix \[ P^2 \]. We now have \[ P^1 a = P^2 P^3 a \] for each \[ a \in E \] and therefore \[ P^1 = P^2 P^3 \], thus establishing the conjecture.

It can readily be shown that if \[ P^1 < P^3 \], then for each \( j = 1, 2, \ldots, n \)

\[
\begin{pmatrix}
\hat{p}_{1j} \\
\hat{p}_{2j} \\
\vdots \\
\hat{p}_{nj}
\end{pmatrix}
<
\begin{pmatrix}
\hat{p}_{1j} \\
\hat{p}_{2j} \\
\vdots \\
\hat{p}_{nj}
\end{pmatrix}.
\]

It would be interesting to establish the converse.

**Reference**


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