ON A CONJECTURE CONCERNING DOUBLY STOCHASTIC MATRICES

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A doubly stochastic (d.s.) matrix is a real $n\times n$ matrix $P=(p_{ij})$ such that

$$p_{ij} \geqq 0, \qquad 1 \leqq i \leqq n, 1 \leqq j \leqq n, \tag{1}$$

$$\sum_j p_{ij} = 1, \qquad 1 \leqq i \leqq n, \tag{2}$$

and

$$\sum_i p_{ij} = 1, \qquad 1 \leqq j \leqq n. \tag{3}$$

We introduce a partial order among d.s. matrices by defining

$$P^1 \prec P^3 \tag{4}$$

to mean there exists a d.s. matrix P^2 such that

$$P^1 = P^2P^3. \tag{5}$$

We introduce a partial order among real vectors $a=(a_1, \cdots, a_n)$ of our real n-dimensional space E by defining

$$a \prec b \tag{6}$$

to mean for each real convex ϕ

$$\sum_i \phi(a_i) \leqq \sum_i \phi(b_i). \tag{7}$$

By [HLP, p. 89], $a\prec b$ if and only if there exists a d.s. matrix P such that

$$a = Pb = \left(\sum_j p_{1j}b_j, \cdots, \sum_j p_{nj}b_j \right).$$

This implies that for each real n-vector a,

$$P^1 \prec P^3 \rightarrow P^1a \prec P^3a. \tag{8}$$

Kakutani has raised the following conjecture.

Conjecture. If, for each real n-vector a, $P^1a\prec P^3a$, then $P^1\prec P^3$. By [HLP, p. 89] if the hypothesis is satisfied there exists a d.s.

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matrix P_a^2 such that

$$P^1 a = P_a^2 P^3 a. \tag{9}$$

The issue is to show that if there exists such a P_a^2 for each vector a, there exists a d.s. P^2 independent of a such that $P^1 = P^2 P^3$.

Let $\mathfrak{P}$ be the collection of vectors with non-negative components. Requirement (1) is equivalent to the requirement

$$P\mathfrak{P} \subset \mathfrak{P}. \tag{10}$$

If e is the vector all of whose components are unity, then requirement (2) is equivalent to requirement

$$Pe = e, \tag{11}$$

i.e., e is a characteristic vector of characteristic value unity. If $\bar{e}$ is the element of $\overline{E}$, the conjugate space of E, whose value at $a \in E$ is given by $(\bar{e}, a) = \sum_i a_i$, then $\bar{e}^\perp = \{a \mid (\bar{e}, a) = 0\}$ is the set of vectors $a \in E$ whose components add up to zero. Requirement (3) is equivalent to

$$P(\bar{e}^\perp) \subset \bar{e}^\perp. \tag{12}$$

PROOF OF CONJECTURE. Suppose now $P^1 a \prec P^3 a$ for each $a \in E$. Consider a mapping ψ: $P^3 E \to P^1 E$ defined as follows: if $b \in P^3 E$, for some $a \in E$ we have $b = P^3 a$, then let $\psi(b) = P^1 a$. We first prove (i) that we have a valid definition, i.e., $\psi(b)$ is uniquely defined by the above and then we prove (ii) that ψ: $P^3 E \to P^1 E$ is a linear transformation.

In order to prove (i) suppose that $P^3 a' = P^3 a'' = b$. We wish to show that $P^1 a' = P^1 a''$. If $P^3 a' = P^3 a''$, then $P^3(a' - a'') = 0$, the zero vector. Since $P^1(a' - a'') \prec P^3(a' - a'')$, by (9) we deduce that $P^1(a' - a'') = 0$ and so $P^1 a' = P^1 a''$. Thus we have shown that ψ is uniquely defined. The linearity (ii) of ψ is now trivial. If α is a real scalar and $P^3 a = b$, then $P^3 \alpha a = \alpha b$ and $\psi(\alpha b) = P^1 \alpha a = \alpha P^1 a = \alpha \psi(b)$. Also if $P^3 a' = b'$ and $P^3 a'' = b''$, then $P^3(a' + a'') = b' + b''$ and $\psi(b' + b'') = P^1(a' + a'') = P^1 a' + P^1 a'' = \psi(b') + \psi(b'')$.

Suppose $P^3 a \in \mathfrak{P}$. Since $P^1 a \prec P^3 a$ by (9) there exists a d.s. P_a such that $P^1 a = P_a P^3 a \in \mathfrak{P}$. Thus $\psi(\mathfrak{P} \cap P^3 E) \subset \mathfrak{P}$. Since P^1 and P^3 are d.s. matrices, $P^1 e = P^3 e = e$ and so $\psi(e) = e$. If by $(Pa)_i$ we denote the ith component of Pa, then by [HLP, p. 89] and the assumption $P^1 a \prec P^3 a$ we have

$$\sum_i (P^1 a)_i = \sum_i (P^3 a)_i.$$

In particular

$$\psi(\bar{e}^{\perp} \cap P^3E) \subset \bar{e}^{\perp}.$$

We can now extend ψ to a function Ψ on all of E by letting $\Psi(b) = 0$ for each $b \in E \cap (P^3E)'$, the complement of P^3E, and $\Psi(b) = \psi(b)$ for each $b \in P^3E$. Now Ψ is a linear transformation satisfying the requirements

$$\Psi(\mathfrak{P}) \subset \mathfrak{P}, \qquad \Psi(e) = e, \qquad \Psi(\bar{e}^{\perp}) \subset \bar{e}^{\perp}.$$

Therefore we can represent Ψ by a d.s. matrix P^2. We now have $P^1a = P^2P^3a$ for each $a \in E$ and therefore $P^1 = P^2P^3$, thus establishing the conjecture.

It can readily be shown that if $P^1 < P^3$, then for each $j = 1, 2, \cdots, n$

$$(\overset{1}{p}_{1j}, \overset{1}{p}_{2j}, \cdots, \overset{1}{p}_{nj}) < (\overset{3}{p}_{1j}, \cdots, \overset{3}{p}_{nj}).$$

It would be interesting to establish the converse.

Reference

G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, 1934.

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