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THE INSTITUTE FOR ADVANCED STUDY AND
THE UNIVERSITY OF PENNSYLVANIA

DIMENSION AND DISCONNECTION

BYRON H. MCCANDLESS

Let X be a semi-compact separable metric space. We shall prove the following theorem using results found in Hurewicz and Wallman's book *Dimension theory* (Princeton University Press, 1948):

THEOREM. dim $X \leqq n$ *if and only if any closed subset of* X *containing at least two points can be disconnected by a closed set of dimension* $\leqq n-1$.

The necessary and sufficient condition stated in the theorem was found in looking for an n-dimensional analogue of the property of a space being totally disconnected (property α_0 below) and will be denoted by α_n.

Hurewicz and Wallman show (p. 20) that the following three properties of the space X are equivalent:

α_0. X is totally disconnected.

β_0. Any two points in X can be separated.

γ_0. Any point can be separated from a closed set not containing it, that is, dim $X=0$.

They also show (p. 36) that the following n-dimensional analogues of β_0 and γ_0 are equivalent:

β_n. Any two points in X can be separated by a closed set of dimension $\leqq n-1$.

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γ_n. Any point can be separated from a closed set not containing it by a closed set of dimension $\leqq n-1$, that is, dim $X \leqq n$.

As we have already noted, the n-dimensional analogue of α_0 is:

α_n. Any closed subset of X containing at least two points can be disconnected by a closed set of dimension $\leqq n-1$.

Obviously γ_n implies β_n and β_n implies α_n. We shall show that α_n implies γ_n. It will then follow, in analogy with the 0-dimensional case, that α_n, β_n, and γ_n are equivalent, thus proving the theorem. The known equivalence of β_n and γ_n is not used in our proof.

We are to show that if the space X possesses property α_n, then dim $X \leqq n$. Since X is the countable union of compact sets we need only to show, by virtue of the sum theorem for dimension n (p. 30), that this is true of a compact space. Therefore, from this point on, let X denote a compact separable metric space. The method of the following proof is due essentially to Hurewicz and Wallman. Let C be a closed subset of X and f a mapping of C in the n-sphere S_n; it suffices to show (p. 83) that f can be extended over X. Suppose, to the contrary, that f cannot be extended over X. There then exists (p. 94) a closed set K such that:

(1) f cannot be extended over $C \cup K$, but

(2) f can be extended over $C \cup K'$ where K' is any proper closed subset of K.

(This statement is false for spaces which are only locally compact.) If K contains at most one point, f clearly can be extended over $C \cup K$ in contradiction to (1). Therefore K contains at least two points. It then follows from property α_n that there exist (p. 47) two proper closed subsets K_1 and K_2 of K such that $K_1 \cup K_2 = K$ and dim $K_1 \cap K_2 \leqq n-1$. By (2), f can be extended to mappings f_1 and f_2 over $C \cup K_1$ and $C \cup K_2$ respectively. Since dim $K_1 \cap K_2 \leqq n-1$, each of the extensions f_1 and f_2 can be extended (p. 88) over the union $C \cup K$ of $C \cup K_1$ and $C \cup K_2$. Therefore f can be extended over $C \cup K$ in contradiction to (1). This contradiction proves that f can, in fact, be extended over X. Consequently dim $X \leqq n$, as was to be proved.

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