Let $X$ be a semi-compact separable metric space. We shall prove the following theorem using results found in Hurewicz and Wallman's book *Dimension theory* (Princeton University Press, 1948):

**Theorem.** $\dim X \leq n$ if and only if any closed subset of $X$ containing at least two points can be disconnected by a closed set of dimension $\leq n - 1$.

The necessary and sufficient condition stated in the theorem was found in looking for an $n$-dimensional analogue of the property of a space being totally disconnected (property $\alpha_0$ below) and will be denoted by $\alpha_n$.

Hurewicz and Wallman show (p. 20) that the following three properties of the space $X$ are equivalent:

- $\alpha_0$. $X$ is totally disconnected.
- $\beta_0$. Any two points in $X$ can be separated.
- $\gamma_0$. Any point can be separated from a closed set not containing it, that is, $\dim X = 0$.

They also show (p. 36) that the following $n$-dimensional analogues of $\beta_0$ and $\gamma_0$ are equivalent:

- $\beta_n$. Any two points in $X$ can be separated by a closed set of dimension $\leq n - 1$.

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Any point can be separated from a closed set not containing it by a closed set of dimension \( \leq n-1 \), that is, \( \dim X \leq n \).

As we have already noted, the \( n \)-dimensional analogue of \( \alpha_n \) is:
\( \alpha_n \). Any closed subset of \( X \) containing at least two points can be disconnected by a closed set of dimension \( \leq n-1 \).

Obviously \( \gamma_n \) implies \( \beta_n \) and \( \beta_n \) implies \( \alpha_n \). We shall show that \( \alpha_n \) implies \( \gamma_n \). It will then follow, in analogy with the 0-dimensional case, that \( \alpha_n, \beta_n, \) and \( \gamma_n \) are equivalent, thus proving the theorem. The known equivalence of \( \beta_n \) and \( \gamma_n \) is not used in our proof.

We are to show that if the space \( X \) possesses property \( \alpha_n \), then \( \dim X \leq n \). Since \( X \) is the countable union of compact sets we need only to show, by virtue of the sum theorem for dimension \( n \) (p. 30), that this is true of a compact space. Therefore, from this point on, let \( X \) denote a compact separable metric space. The method of the following proof is due essentially to Hurewicz and Wallman. Let \( C \) be a closed subset of \( X \) and \( f \) a mapping of \( C \) in the \( n \)-sphere \( S_n \); it suffices to show (p. 83) that \( f \) can be extended over \( X \). Suppose, to the contrary, that \( f \) cannot be extended over \( X \). There then exists (p. 94) a closed set \( K \) such that:

1. \( f \) cannot be extended over \( C \cup K \), but
2. \( f \) can be extended over \( C \cup K' \) where \( K' \) is any proper closed subset of \( K \).

(This statement is false for spaces which are only locally compact.) If \( K \) contains at most one point, \( f \) clearly can be extended over \( C \cup K \) in contradiction to (1). Therefore \( K \) contains at least two points. It then follows from property \( \alpha_n \) that there exist (p. 47) two proper closed subsets \( K_1 \) and \( K_2 \) of \( K \) such that \( K_1 \cup K_2 = K \) and \( \dim K_1 \cap K_2 \leq n-1 \). By (2), \( f \) can be extended to mappings \( f_1 \) and \( f_2 \) over \( C \cup K_1 \) and \( C \cup K_2 \) respectively. Since \( \dim K_1 \cap K_2 \leq n-1 \), each of the extensions \( f_1 \) and \( f_2 \) can be extended (p. 88) over the union \( C \cup K \) of \( C \cup K_1 \) and \( C \cup K_2 \). Therefore \( f \) can be extended over \( C \cup K \) in contradiction to (1). This contradiction proves that \( f \) can, in fact, be extended over \( X \). Consequently \( \dim X \leq n \), as was to be proved.

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