A NOTE ON BERNOULLI NUMBERS AND POLYNOMIALS OF HIGHER ORDER

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1. Introduction. Following the notation of Nörlund [5, Chap. 6], we defined $B_m^{(k)}$, $B_m^{(k)}(u)$ by means of

$$(1.1)\qquad \left(\frac{x}{e^x-1}\right)^k e^{xu} = \sum_{m=0}^{\infty} B_m^{(k)}(u)\,\frac{x^m}{m!},\quad B_m^{(k)} = B_m^{(k)}(0)\ (k \geqq 1).$$

In the present paper we prove a number of theorems concerning $B_m^{(k)}(u)$. It will be convenient to employ the abbreviations

$$(1.2)\qquad \begin{aligned} (m)_k &= m(m-1)\cdots(m-k+1), & (m)_0 &= 1,\\ [m]_k &= (a^m-1)(a^{m-1}-1)\cdots(a^{m-k+1}-1), & [m]_0 &= 1. \end{aligned}$$

In the following theorems p denotes an odd prime; the rational numbers a, u are integral (mod p) and $p\nmid a$. We now state the following theorems.

THEOREM 1. *The number*

$$(1.3)\qquad U_m^{(k)} = [m]_k B_m^{(k)}(u)/(m)_k \qquad (m \geqq k \geqq 1)$$

is integral (mod p).

THEOREM 2. *If $k<p-1$, $m\not\equiv 0, 1, \cdots, k-1$* (mod $p-1$), *$m\geqq k\geqq 1$, then $B_m^{(k)}(u)/(m)_k$ is integral* (mod p). *In particular $B_m^{(k)}(u)$ is integral* (mod p).

THEOREM 3. *If $k<p-1$, $m\not\equiv 0, 1, \cdots, k-1$* (mod $p-1$), *$m\geqq k\geqq 1$, $p^r\mid (m)_k$, then the numerator of $B_m^{(k)}(u)$ is divisible by p^r.*

THEOREM 4. *Let $U_m^{(k)}$ have the same meaning as in* (1.3). *If $(p-1)p^{e-1}\mid b$, $m\geqq rb+k$, $k\geqq 1$, then*

$$(1.4)\qquad \sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}U_{m+sb}^{(k)} \equiv 0 \pmod{p^{re}}.$$

THEOREM 5. *Put*

$$(1.5)\qquad T_m^{(k)} = B_m^{(k)}(u)/(m)_k \qquad (m \geqq k \geqq 1).$$

If $k<p-1$, $m\not\equiv 0, 1, \cdots, k-1$ (mod $p-1$), *$m\geqq rb+k$, then*

Presented to the Society, February 23, 1952; received by the editors November 21, 1951.

$$\sum_{s=0}^{r} (-1)^{r-s}\binom{r}{s} T_{m+sb}^{(k)} \equiv 0 \pmod{p^{re}}. \tag{1.6}$$

THEOREM 6. *If* $k<p-1$, $m\not\equiv 0, 1, \cdots, k-1 \pmod{p-1}$, $m\geqq rb+k$, $r\geqq k$, *then*

$$\sum_{s=0}^{r} (-1)^{r-s}\binom{r}{s} B_{m+sb}^{(k)}(u) \equiv 0 \pmod{p^{(r-k)e}}. \tag{1.7}$$

THEOREM 7. *If* $k\leqq p-1$, $m\equiv s_0 \pmod{p-1}$, $0\leqq s_0\leqq k-1$, *then*

$$pB_m^{(k)}(u) \equiv \frac{(-1)^{k-s_0}}{(k-1)!}\,\frac{(m)_k}{m-s_0}\binom{k-1}{s_0} B_{s_0}^{(k)}(u) \pmod{p}. \tag{1.8}$$

THEOREM 8. *Let* $m\equiv s_0 \pmod{p-1}$, $0\leqq s_0<p-1$. *If* $s_0\neq 0$, *then*

$$pB_m^{(p)}(u) \equiv \frac{(m)_p}{m-s_0}\,u^{s_0} \pmod{p}; \tag{1.9}$$

in particular if $p\,|\,m-s_0$, *then*

$$pB_m^{(p)}(u) \equiv -u^{s_0}.$$

However, if $s_0=0$, *then*

$$pB_m^{(p)}(u) \equiv (m)_p\left(\frac{1}{m} + \frac{u^{p-1}-1}{m-p+1}\right) \pmod{p}; \tag{1.10}$$

in particular if $p\,|\,m$, *then* $pB_m^{(p)}(u)\equiv -1$, *if* $p\,|\,m+1$, *then* $pB_m^{(p)}(u)$ $\equiv 1-u^{p-1}$.

For references in the case $k=1$, see [**1**, Chap. 1; **2**; **3**; **4**, Chap. 14; **6**]. Vandiver [**6**] has also discussed the case $k=2$; indeed his numbers of the second order are somewhat more general.

2. **Proof of Theorem 1.** Let $\eta(x)$ denote a (formal) power series of the type

$$1 + \sum_{1}^{\infty} c_m(e^x-1)^m, \tag{2.1}$$

where the c_m are integral (mod p). Put

$$g(x) = \left(\frac{x}{e^x-1}\right)^k \eta(x). \tag{2.2}$$

If for brevity we define $\delta^r g(x)$ recursively by means of

$$\delta g(x) = g(ax) - g(x), \qquad \delta^{r+1}g(x) = \delta^r g(ax) - a^r\delta^r g(x),$$

then in the first place, we have

$$\delta g(x) = \left(\frac{ax}{e^{ax}-1}\right)^k \eta(ax) - \left(\frac{x}{e^x-1}\right)^k \eta(x) = \frac{x^k}{(e^x-1)^{k-1}}\eta_1(x),$$

as is easily verified; here $\eta_1(x)$ represents a series of the form (2.1). At the next step we find

$$\begin{aligned}\delta^2 g(x) &= \frac{a^k x^k}{(e^{ax}-1)^{k-1}}\eta_1(ax) - \frac{ax^k}{(e^{ax}-1)^{k-1}}\eta_1(x)\\ &= \frac{x^k}{(e^x-1)^{k-2}}\eta_2(x),\end{aligned}$$

where $\eta_2(x)$ is also of the form (2.1). Continuing in this way, we finally get

$$\delta^k g(x) = x^k \eta_k(x), \tag{2.3}$$

where of course $\eta_k(x)$ is of the form (2.1). Now let $\eta(x)=e^{xu}$ in (2.2); then it is clear from (1.1) that

$$\frac{\delta^k g(x)}{x^k} = \sum_{m=k}^{\infty} \frac{[m]_k B_m^{(k)}(u)}{(m)_k} \frac{x^{m-k}}{(m-k)!} = \sum_{m=0}^{\infty} U_{m+k}^{(k)} \frac{x^m}{m!}. \tag{2.4}$$

Now on the other hand it follows immediately from (2.1) that

$$\eta(x) = \eta_k(x) = \sum_{n=0}^{\infty} b_n x^n/n!,$$

where the b_n are integral (mod p). Comparison with (2.3) and (2.4) yields the theorem.

3. **Proof of Theorems 2 and 3.** Suppose now that a is a primitive root (mod p); then it is clear from the hypothesis of Theorem 2 that none of the factors $a^{k-i}-1$, $i=0, 1, \cdots, k-1$, is divisible by p. Consequently $[m]_k$ is prime to p and thus Theorem 1 implies Theorem 2.

In the next place, let $p^r \mid (m)_k$. Since, as we have just seen, $p \nmid [m]_k$, it follows from (1.3) that $B_m^{(k)}(u) \equiv 0 \pmod{p^r}$. Hence Theorem 3 follows.

4. **Proof of Theorem 4.** We note first that for $\eta(x)$ as defined by (2.1), we have

$$\eta(x) = 1 + \sum_{t=1}^{\infty} c_t \sum_{s=0}^{t} (-1)^{t-s}\binom{t}{s} \sum_{m=0}^{\infty} \frac{s^m x^m}{m!}.$$

Hence if we put

$$\eta(x) = 1 + \sum_{m=1}^{\infty} d_m x^m/m!,$$

it follows that

$$d_m = \sum_{t=1}^{n} c_t \sum_{s=0}^{t} (-1)^{t-s}\binom{t}{s} s^m \qquad (n \geqq m), \tag{4.1}$$

since the inner sum in the right member of (4.1) vanishes for $n > m$. Then clearly

$$\sum_{j=0}^{r} (-1)^{r-j}\binom{r}{j} d_{m+sb} = \sum_{t=1}^{\infty} c_t \sum_{s=0}^{t} (-1)^{t-s}\binom{t}{s}(s-1)^r s^m,$$

where of course the outer sum in the right member is finite. It follows at once that

$$\sum_{j=0}^{r} (-1)^{r-j}\binom{r}{j} d_{m+jb} \equiv 0 \pmod{p^{re}} \tag{4.2}$$

provided $m \geqq rb$.

Turning now to $U_m^{(k)}$, we get from (2.3) and (2.4) that $\delta^k g(x)/x^k$ is of the form $\eta(x)$ and that the general term in the expansion is of the form $U_{m+k}^{(k)} x^m/m!$ $(m \geqq 0)$. Thus we may take $d_m = U_{m+k}^{(k)}$, and (4.1) and (4.2) apply. In particular (4.2) implies

$$\sum_{j=0}^{r} (-1)^{r-j}\binom{r}{j} U_{m+k+jb}^{(k)} \equiv 0 \pmod{p^{re}} \tag{4.3}$$

provided $m \geqq rb$. If we replace $m+k$ by m, it is clear that Theorem 4 holds.

5. **Proof of Theorem 5.** If we substitute from (1.3) in (4.3), we get

$$\sum_{j=0}^{r} (-1)^{r-j}\binom{r}{j} \frac{[m+jb]_k B_{m+jb}^{(k)}(u)}{(m+jb)_k} \equiv 0 \pmod{p^{re}} \tag{5.1}$$

provided $m \geqq rb+k$. Suppose now that a is a primitive root (mod p) such that $a^{p-1} \equiv 1 \pmod{p^w}$ for an arbitrarily assigned w. By Theorem 2 we know that $B_{m+jb}^{(k)}(u)/(m+jb)_k$ is integral. Hence it suffices to take $w = re$, so that

$$[m+jb]_k \equiv [m]_k \pmod{p^{re}} \qquad (j = 0, 1, \cdots, r).$$

Thus the left member of (5.1) is congruent to

$$[m]_k \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} B_{m+jb}^{(k)}(u)/(m+jb)_k \pmod{p^{re}}.$$

Since $p \nmid [m]_k$, (1.6) follows immediately.

6. **Proof of Theorem 6.** We make use of a device employed by Nielsen [**2**, Chap. 14]. Let

$$A_{r,q} = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \binom{m+sb}{q} T_{m+sb}^{(k)}, \tag{6.1}$$

so that $A_{r,0}$ denotes the left member of (1.6) and $A_{r,k}$ the left member of (1.7). We require the recursion

$$(m + rb - q)A_{r,q} + rbA_{r-1,q} = (q+1)A_{r,q+1}, \tag{6.2}$$

which is easily verified by substituting from (6.1). Now by the last theorem $A_{r,0} \equiv 0 \pmod{p^{re}}$; hence repeated application of (6.2) leads to

$$A_{r,q} \equiv 0 \pmod{p^{(r-q)e}} \tag{6.3}$$

provided $q \leqq r$, $q < p$. In particular if we take $q = k$ in (6.3), Theorem 6 follows at once.

7. **Proof of Theorems 7 and 8.** We shall require the following formula [**5**, p. 148, (87)]:

$$B_m^{(k)}(u) = k \binom{m}{k} \sum_{s=0}^{k-1} (-1)^{k-1-s} \binom{k-1}{s} \frac{B_{m-s}(u)}{m-s} B_s^{(k)}(u), \tag{7.1}$$

where $B_m(u) = B_m^{(1)}(u)$; we also need

$$pB_m(u) \equiv \begin{cases} -1 \\ 0 \end{cases} \pmod{p} \qquad \begin{matrix} (p-1 \mid m), \\ (p-1 \nmid m). \end{matrix} \tag{7.2}$$

Now let $m \equiv s_0 \pmod{p-1}$, where $0 \leqq s_0 \leqq k-1$. Since for $s < k$

$$B_s^{(k)}(u) = \frac{s!}{(k-1)!} \left(\frac{d}{du}\right)^{k-1-s} (u-1)(u-2) \cdots (u-k+1), \tag{7.3}$$

it is clear that $B_s^{(k)}(u)$ is integral (mod p). Thus if we apply (7.2) to the right member of (7.1), we get

$$pB_m^{(k)} \equiv (-1)^{k-s_0} k \binom{m}{k} \binom{k-1}{s_0} \frac{B_{s_0}^{(k)}(u)}{m-s_0} \pmod{p},$$

which is the same as (1.8).

To prove Theorem 8, we again use (7.1). Then for $k=p$, $s_0\neq 0$, it is clear that (7.1) and (7.2) imply

$$pB_m^{(p)}(u) \equiv (-1)^{s_0+1} \frac{(m)_p}{(p-1)!}\binom{p-1}{s_0}\frac{B_{s_0}^{(p)}}{m-s_0} \pmod{p}.$$

Now

$$\binom{p-1}{s} \equiv (-1)^s$$

and by (7.3)

$$B_s^{(p)}(u) \equiv \frac{s!}{(p-1)!}\left(\frac{d}{du}\right)^{p-1-s}(u^{p-1}-1) \equiv u^s \qquad (s \leqq p-1).$$

Thus

$$pB_m^{(p)}(u) \equiv \frac{(m)_p}{m-s_0} u^{s_0} \pmod{p},$$

which is identical with (1.9).

As for the case $s_0=0$, the only difference is that there are now two terms in (7.1) to consider, namely, those corresponding to $s=0$, $s=p-1$. Thus

$$(7.4) \qquad pB_m^{(p)}(u) \equiv -\frac{(m)_p}{(p-1)!}\left(\frac{1}{m}+\frac{1}{m-p+1}B_{p-1}^{(p)}(u)\right);$$

but by (7.3)

$$B_{p-1}^{(p)}(u) = (u-1)(u-2)\cdots(u-p+1) \equiv u^{p-1}-1.$$

Substitution in (7.4) yields (1.10).

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