THE QUADRIC OF LIE

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1. **Introduction.** In this paper we describe a simple way of generating the quadric of Lie. Similar methods of generating the quadrics of Lane [**7**], of Wilczynski [**8**] and Fubini, and the quadrics $Q(l_2, m_2, l_3, m_3)$ described previously by us [**3**] are described. In the description of the method of generating these quadrics another description of the curves corresponding to the developables of reciprocal congruences is found.

Let the homogeneous projective coordinates (x^1, x^2, x^3, x^4) of a generic point x of a surface S be functions of the asymptotic parameters u, v so normalized [**4**] that they satisfy the differential equations

$$
\begin{aligned}
x_{uu} &= \theta_u x_u + \beta x_v + px, \\
x_{vv} &= \gamma x_u + \theta_v x_v + qx, \qquad \theta = \log R.
\end{aligned} \tag{1.1}
$$

We have called the line determined by the points $x_u x_v$ the R-harmonic line, and that determined by x, x_{uv} the R-conjugate line.

If we define the homogeneous projective coordinates ξ of the tangent plane by the expression

$$\xi = R^{-1}(x, x_u, x_v),$$

it follows that the functions ξ satisfy the differential equations [**6**]

$$
\begin{aligned}
\xi_{uu} &= \theta_u \xi_u - \beta \xi_v + \pi x, \\
\xi_{vv} &= -\gamma \xi_u + \theta_v \xi_v + \chi x,
\end{aligned}
$$

wherein

$$\pi = p + \beta_v + \beta\theta_v, \qquad \chi = q + \gamma_u + \gamma\theta_u.$$

It is well known [**2**] that a geometric characterization can be given for reciprocal lines associated with S at x without using the quadrics of Darboux with respect to which the lines are reciprocal. Let a line l_2 in the tangent plane to S at x be determined by the points r, s whose coordinates are defined by the expressions

$$r = x_u - bx, \qquad s = x_v - ax. \tag{1.2}$$

A point r_v on the tangent to the locus of r as x generates the asymp-

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totic curve u=const. has coordinates

$$r_v = x_{uv} - bs - (ab + b_v)x.$$

Similarly the point s_u with coordinates

$$s_u = x_{uv} - ar - (ab + a_u)x$$

is found.

The common intersector l_1 of the tangents to the loci of the points r and s which passes through x intersects these respective tangents in the points z_1, z_2 defined by the expressions

$$\begin{aligned} z_1 &= x_{uv} - ax_u - bx_v + (ab - b_v)x, \\ z_2 &= x_{uv} - ax_u - bx_v + (ab - a_u)x. \end{aligned} \tag{1.3}$$

The lines l_1, l_2 are reciprocal lines. Moreover *the quadrics of Darboux may be characterized as the only quadrics through x having l_1, l_2 as reciprocal lines for all lines l_2 in the tangent plane to S at x.*

The harmonic conjugate h of x with respect to the points z_1, z_2 has coordinates

$$h = x_{uv} - ax_u - bx_v + [ab - 2^{-1}(a_u + b_v)]x.$$

We shall call *h the harmonic point* on l_1. The points z_1, z_2 (and h) coincide if and only if $a_u - b_v = 0$, that is, if and only if the congruence Γ_2 of lines l_2 is central [1] to S.

2. **A family of projectivities on l_2.** Let z be any point on l_1 (except x) with coordinates of the form

$$z = x_{uv} - ax_u - bx_v + \phi x. \tag{2.1}$$

As x generates the curve C through x of the one-parameter family of curves defined by

$$\mu du - \lambda dv = 0, \tag{2.2}$$

the point z describes a curve. The tangent of this curve intersects the tangent plane to S at x in a point whose projection $\bar{y}$ from x on the line l_2 has coordinates

$$\bar{y} = \bar{\lambda} r + \bar{\mu} s$$

wherein

$$\begin{aligned} \bar{\lambda} &= (\kappa - ab - a_u + \phi)\lambda + (\chi - a_v - b\gamma - a^2)\mu, \\ \bar{\mu} &= (\pi - b_u - a\beta - b^2)\lambda + (\kappa - ab - b_v + \phi)\mu, \\ \kappa &= \beta\gamma + \theta_{uv}. \end{aligned} \tag{2.3}$$

The tangent to C at x intersects l_2 in a point y with coordinates

$$y = \lambda r + \mu s.$$

Formulas (2.3) therefore represent a family $F(\phi)$ of projectivities on l_2, y and $\bar{y}$ being corresponding points in any particular member of the family.

In $F(\phi)$ there is one involution determined by

$$\phi = ab + 2^{-1}(a_u + b_v) - \kappa. \tag{2.4}$$

The point I, defined by (2.1) and (2.4), will be called *the involutory point on* l_1. Its coordinates I_1 are defined by the expression

$$I_1 = x_{uv} - ax_u - bx_v + [ab + 2^{-1}(a_u + b_v) - \kappa]x. \tag{2.5}$$

A comparison of (2.5) with the coordinates of the focal points [6] on l_1 shows that *the involutory point on l_1 is the harmonic conjugate of x with respect to the focal points.*

Let g be the harmonic conjugate of x with respect to the harmonic point h, and the involutory point I_1. The coordinates of g are readily found to be given by the formula

$$g = x_{uv} - ax_u - bx_v + (ab - 2^{-1}\kappa)x. \tag{2.6}$$

Defining the local coordinates (x_1, x_2, x_3, x_4) of a point X by the expression

$$X = x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uv},$$

we find that the locus of the point g as l_1 varies in the tangent plane has the equation

$$x_2 x_3 - x_1 x_4 - 2^{-1}(\beta\gamma + \theta_{uv})x_4^2 = 0.$$

That is, *the locus of g is the quadric of Lie of S at x. The lines (gr) and (gs) are generators of this quadric, and hence the quadric of Lie is enveloped by the plane of g and l_2.* We shall call the point g *the generating point of l_1.*

Returning to the family $F(\phi)$ of projectivities defined by (2.3) we observe that *the double points of these projectivities are identical for all members of the family.* These double points are given by the quadratic equation

$$(\pi - b_u - a\beta - b^2)\lambda^2 - (b_v - a_u)\lambda\mu - (\chi - a_v - b\gamma - a^2)\mu^2 = 0. \tag{2.7}$$

That is, the double points of the projectivities of $F(\phi)$ are the intersections of the tangents to the Γ_1-curves of the congruence Γ_1 of lines l_1 with l_2. Hence another interpretation may be placed on the curves corresponding to the developables of a congruence Γ_1 of lines protruding from a surface.

The above constructions may be dualized. Any plane ζ through l_2 has coordinates of the form

$$\zeta = \xi_{uv} - a\xi_u - b\xi_v + \phi\xi.$$

As x moves along the curve C defined by (2.2), the plane ζ envelops a developable surface whose generator in ζ determines with x a plane whose line of intersection with the tangent plane ξ and l_1 determine a plane whose coordinates are $\bar{\lambda}\rho+\bar{\mu}\sigma$ wherein $\rho=\xi_u-b\xi$, $\sigma=\xi_v-a\xi$ and

$$(2.8)\qquad \begin{aligned} \bar{\lambda} &= (\kappa + ab - a_u + \phi)\lambda + (q - a_v + b\gamma - a^2)\mu, \\ \bar{\mu} &= (p - b_u + a\beta - b^2)\lambda + (\kappa + ab - b_v + \phi)\mu. \end{aligned}$$

The double planes of the family (2.8) *of projectivities correspond to the developables of the congruence* Γ_2 *of lines* l_2.

Among the projectivities (2.8) is one involution determined by the plane through l_2 with coordinates

$$\xi_{uv} - a\xi_u - b\xi_v + [ab + 2^{-1}(a_u + b_v) - \kappa]\xi.$$

We shall call this plane *the involutory plane* through l_2. The involutory plane intersects l_1 in a point I_2 whose coordinates have the form

$$I_2 = x_{uv} - ax_u - bx_v + [ab - 2^{-1}(a_u + b_v)]x.$$

The harmonic conjugate of x *with respect to the involutory point* I_1 *and the point* I_2 *is the generating point* g.

Similarly *an enveloping plane* (the dual of the generating point g) may be described. Its coordinates are

$$\xi_{uv} - a\xi_u - b\xi_v + (ab - 2^{-1}\kappa)\xi.$$

The envelope of this enveloping plane is of course the quadric of Lie.

3. **Quadrics associated with the surface.** Some of the notions developed in the paper may be used to describe methods of generating other quadrics associated with a surface at a point. Before describing these methods, we first give an extension of the R_λ-associate of a line l_2 and the R_λ-derived line and curves developed by Bell.

Let there be given on S two one-parameter families of curves, not necessarily distinct. The differential equations of such a set of curves may be written in the form

$$(3.1)\qquad (dv - \lambda du)(du - \mu dv) = 0.$$

Denote the curves of these families through x by C_λ, C_μ respectively. As x generates the curve C_λ the point r defined by (1.2) describes a curve. Similarly the point s describes a curve as x moves on C_μ. The common intersector $l_{1\lambda\mu}$ through x of the tangents to these loci of r and s joins x to the point z whose coordinates are

$$z = x_{uv} - (a - \gamma\lambda)x_u - (b - \beta\mu)x_v. \tag{3.2}$$

The reciprocal of $l_{1\lambda\mu}$ is the line $l_{2\lambda\mu}$ joining the points r_μ, s_λ defined by the formulas

$$r_\mu = r + \beta\mu x, \qquad s_\lambda = s + \gamma\lambda x. \tag{3.3}$$

We shall call the line $l_{2\lambda\mu}$ *the* $R_{\lambda\mu}$*-associate of* l_2. If in its definition the curves C_λ and C_μ are identical, the $R_{\lambda\mu}$-associate is the R_λ-associate of l_2 as defined [1] by Bell.

Bell has called [1] the line joining x to the intersection of the R_λ-associate of l_2 with l_2 the R_λ-derived line. Similarly we call the line determined by x and the intersection of the $R_{\lambda\mu}$-associate of l_2 with l_2 *the* $R_{\lambda\mu}$*-derived line.* This line has the direction dv/du defined by

$$\gamma\lambda du - \beta\mu dv = 0. \tag{3.4}$$

We may readily verify that *the tangent to C at x and the $R_{\lambda\mu}$-derived line are conjugate directions if and only if $\mu = -\gamma\lambda^2/\beta$, that is, if and only if the tangent to C_μ is the R_λ-derived line* [1] *of Bell.*

From (2.6) and (3.3) it is readily seen that the generating point $g_{\lambda\mu}$ on $l_{1\lambda\mu}$ has coordinates given by the formula

$$g_{\lambda\mu} = x_{uv} - (a - \gamma\lambda)x_u - (b - \beta\mu)x_v + [(a - \gamma\lambda)(b - \beta\mu) - 2^{-1}\kappa]x.$$

The plane determined by l_2 and $g_{\lambda\mu}$ intersects l_1 (the reciprocal of l_2) in the point $\overset{*}{g}_{\lambda\mu}$ defined by

$$\overset{*}{g}_{\lambda\mu} = x_{uv} - ax_u - bx_v + [ab + \beta\gamma\lambda\mu - 2^{-1}\kappa]x.$$

The locus of $\overset{*}{g}_{\lambda\mu}$ as l_2 varies in the tangent plane to S at x is the quadric $Q_{\lambda\mu}$ whose equation is

$$x_2x_3 - x_1x_4 - 2^{-1}[(1 - h)\beta\gamma + \theta_{uv}]x_4^2 = 0, \qquad h = 2\lambda\mu. \tag{3.5}$$

One of the cross ratios of the asymptotic tangents and the tangents to C_λ, C_μ is $\lambda\mu$. If we impose the condition that this cross ratio be constant, the quadrics $Q_{\lambda\mu}$ becomes the quadrics [7] of Lane. *If either one or both of C_λ, C_μ is an asymptotic curve, the quadric $Q_{\lambda\mu}$ is the quadric of Lie. If the tangents to C_λ and $v=$*const. *separate the tangents to C_μ and $u=$*const. *harmonically, $Q_{\lambda\mu}$ is the quadric of*

Wilczynski. If C_λ, C_μ are curves of a conjugate net, the quadric $Q_{\lambda\mu}$ could be called *the conjugate quadric of* S at x. Its equation is

$$x_2x_3 - x_1x_4 - 2^{-1}[3\beta\gamma + \theta_{uv}]x_4^2 = 0.$$

We have previously [3] characterized a line called the (l_2, m_2, l_3, m_3)-associate of l_2 in terms of constant cross ratios and involving the R_λ-associate and R_λ-derived line of Bell. If we proceed in a similar manner, a generalized (l_2, m_2, l_3, m_3)-associate of l_2 can be characterized using constant cross ratios and replacing the R_λ-associate and R_λ-derived lines by the $R_{\lambda\mu}$-associate and $R_{\lambda\mu}$-derived lines respectively. Such a generalized associate of l_2 is determined by the formulas

$$r_{\lambda\mu} = x_u - (b - k_2)x, \qquad s_{\lambda\mu} = x_v - (a - k_3)x \tag{3.6}$$

wherein

$k_2 = l_2\beta\mu + m_2\gamma\lambda/\mu$, $k_3 = l_3\beta\mu/\lambda + m_3\gamma\lambda$, k, m_2, l_3, m_3 being constants.

Hence *if we use the special* $(0, m_2, l_3, 0)$*-associate* (*or* $(l_2, 0, 0, m_3)$*-associate*) *in place of the* $R_{\lambda\mu}$*-associate in the above characterization of the quadrics* $R_{\lambda\mu}$, *the family of quadrics of Lane with* $h = l_3m_2$ (*or* $h = l_2m_3$) *is obtained.*

Finally it is easy to verify that the plane determined by the R-harmonic line and the generating point on the reciprocal of the above generalized (l_2m_2, l_3m_3)-associate of l_2 intersects the reciprocal of l_2 in the point $g_{\lambda\mu}$ whose coordinates are given by the expression

$$g_{\lambda\mu} = x_{uv} - ax_u - bx_v + [(a - k_3)(b - k_2) - 2^{-1}\kappa]x.$$

The locus of this point is the quadric whose equation is

$$x_2x_3 + x_4[-x_1 + k_2x_2 + k_3x_3 + (k_2k_3 - 2^{-1}\kappa)x_4] = 0.$$

This quadric is a member of a family which might be called *the generalized* (l_2, m_2, l_3, m_3) *quadrics*. In particular if $l_2 = m_2 = l_3 = m_3 = 0$, the quadric is the quadric of Lie.

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TWO NOTES ON NILPOTENT GROUPS

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I

We extend a theorem of Rédei and Szép.[1] Our proof is quite straightforward, and employs a method of considerably more general applicability.[2]

The *lower central series* of a group G is formed by taking $G_1 = G$, and successively defining G_{n+1} to be the commutator (G_n, G). G is *nilpotent* if some $G_{N+1} = 1$. If A and B are subgroups of G, $A \vee B$ is the subgroup generated by the elements of A and of B together, and A^m the subgroup generated by the mth powers of elements of A.

THEOREM. *Let A and K be subgroups of a nilpotent group G, and let $A^{m^e} = 1$ for some integer m^e. Then, for any $n \geq 1$,*

$$(A \vee K)_n = (A^m \vee K)_n \textit{ implies } (A \vee K)_n = K_n.$$

We may clearly suppose that $G = A \vee K$. The elements of G_r can be written as products of commutators of *order r*:

$$(x_1, \cdots, x_r) = ((\cdots((x_1, x_2), x_3) \cdots, x_{r-1}), x_r).$$

Let C_r be the subgroup generated by those commutators for which

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[1] L. Rédei and J. Szép, Monatshefte für Mathematik vol. 55, p. 200. The present proof avoids "counting arguments" and the attendant finiteness conditions; for $n = 1$ the present argument reduces substantially to that of Rédei and Szép. We remark that the hypothesis $A^{m^e} = 1$ admits various modifications.

[2] The basic idea of "expanding" words in commutators of ascending order has been exploited by P. Hall, Proc. London Math. Soc. vol. 36, p. 29; and by O. Grün, J. Reine Angew. Math. vol. 182, p. 158. See also W. Magnus, Monatschefte für Mathematik vol. 47, p. 307, and K. T. Chen, Proceedings of the American Mathematical Society vol. 3, p. 44.