CONTINUOUS COLLECTIONS OF CONTINUOUS CURVES IN THE PLANE

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The purpose of this paper is to consider the class of continuous collections of mutually exclusive compact continuous curves in the plane. Throughout this paper we shall denote by G or G with a subscript or superscript a collection of this class with the property that G with respect to its elements as points is a nondegenerate compact closed point set. G has then the significance of being both a collection of continua and a point set itself.

By a *continuous curve* will be meant a nondegenerate locally connected compact continuum. By a *continuous collection* will be meant a collection which is both upper and lower semi-continuous. By a (simple) *chain* will be meant a finite collection $x_1, x_2, \cdots, x_n$ of open discs (i.e. interiors of simple closed curves) such that $\bar{x}_i \cdot \bar{x}_j$ exists if and only if $|i-j| \leqq 1$ and is the closure of an open disc (i.e. a 2-cell) if it does exist. The x_i are called *links* of the chain. A *subchain* of a chain c is a chain whose links are links of c.

A chain c will be said to *simply cover* a set M if c^* contains M and if for no proper subchain c' of c does the closure of c'^* contain M. Two chains will be said to be mutually exclusive if no link of either intersects any link of the other. A collection C' of sets is said to be a (*closed*) *refinement* of a collection C of sets if (the closure of) each element of C' is a subset of some element of C. An *emanation* point of a continuum M is a point which is the common part of each pair of some three nondegenerate subcontinua of M. A *hereditary continuous curve* is a continuous curve each of whose nondegenerate subcontinua is a continuous curve.

It is immediately clear that if G is connected, G contains uncountably many elements and that only countably many can contain triods [1]. Except for a countable number of elements, each element of G must be either an arc or a simple closed curve. We denote the elements of G which are neither arcs nor simple closed curves by $g_1, g_2, g_3, \cdots$. From the hypothesis of continuity of G it follows immediately that no element of a connected G contains a 2-cell.

THEOREM I. *It is not true that both the collection of arcs and the collection of simple closed curves are dense in G.*

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PROOF. We suppose the contrary. Then there exists an arc t_1, a simple chain c_1 which simply covers t_1 and is of at least eleven links and of link diameter less than one, and two open sets d_1 and d_2 containing the end points of t_1 such that the d_i lie in distinct end links of c_1 and intersect no other links of c_1. We consider the set F of all elements of G lying in c_1^* and intersecting d_1 and d_2. F is open and contains a simple closed curve j_1 as an element. But j_1 contains two mutually exclusive arcs each intersecting d_1 and d_2 and hence every link of c_1. Some arc t_2 of F must then contain two mutually exclusive subarcs each intersecting every link of c_1, as the collection G is continuous, the collection of arcs of G is dense in G, and there exist two mutually exclusive simple chains each a closed refinement of c_1, the sum of the links of each of which intersects every link of c_1 such that, for some open subset F_1 of F containing j_1, every element of F_1 must intersect each link of both chains. By an iteration of the above argument it follows that there exists a sequence of chains $c_1, c_2, \cdots$ such that for each i, c_{i+1} is a closed refinement of c_i, c_i is of link diameter less than $1/i$, c_i^* contains an arc t_i of G intersecting each link of c_i, c_i^* does not contain g_i, and c_{i+1} contains two mutually exclusive simple subchains each of which intersects every link of c_i. $H = c_1^* \cdot c_2^* \cdot c_3^* \cdots$ cannot be an arc or a simple closed curve for H contains uncountably many mutually exclusive continua each of diameter greater than some number. As H is the sequential limiting set of the sequence of elements t_i of G, H must be an element of G, and as H cannot be g_i for any i, it must be an arc or a simple closed curve. Thus Theorem I is established. We note here also that a simple argument could be given to show that H is not even locally connected.

We designate by α either the point at infinity ω or any particular point of the plane.

LEMMA A. *If $f_1, f_2, f_3, \cdots$ is a sequence V of elements of a collection G such that, for each i, f_{i+1} separates f_i from α in the plane, then V has a sequential limiting set k which is an element of G and which either contains α or, for each i, separates f_i from α.*

PROOF. We note that for $i>j$, f_i separates f_j from α. Some subsequence V' of V has a sequential limiting set k' which from the asassumption of continuity of the collection G is an element of G. Let k'' be the sequential limiting set of some other subsequence V'' of V. We wish to show that $k'=k''$ and therefore that k' is the sequential limiting set of V itself. For any element f_{n_j} of V', k' either contains α or separates f_{n_j} from α, for k' is the common part of a monotonic sequence of closed and compact sets each either containing α or separating f_{n_j} from α. If some point P of k' is not a limit point of the sum of the elements of V'', then it readily follows that, for some $i>j$, f_i does not separate f_j from α as no point of k' is separated from α by any element of V. It follows that k' is the sequential limiting set k'' of V'' as V'' has by hypothesis a sequential limiting set. By the argument suggested above for V', $k(=k'=k'')$ must contain α or separate each element of V from α.

LEMMA B. *If g an element of G is a limit element of each of two subsets G_1 and G_2 of G, with G_1^* and G_2^* lying in distinct complementary domains of g in the plane, then g is a simple closed curve.*

PROOF. This lemma follows immediately from the continuity of the collection G, the elementary separation properties of a simple closed curve in the plane, and Theorem[2] 41 on p. 216 of reference [2].

LEMMA C. *If in addition to the hypotheses of Lemma* A *it is also supposed that G is connected and k does not contain α, then any nondegenerate subcontinuum G' of G which contains k and contains no element of G not separated by k from α must contain some term of V.*

PROOF. It follows from Lemma B that G'^*-k and $f_1+f_2+\cdots$ all lie in the same complementary domain of k. But then there exists an arc s in such complementary domain of k from a point of G'^*-k to some point of some f_i such that s does not intersect any f_j, $j>i$. Otherwise s must intersect k. But then f_{i+1} does not separate f_i from α, contrary to hypothesis.

LEMMA D. *If G does not contain any arcs as elements and is connected, then G does not contain two elements such that neither separates the other from ω in the plane.*

PROOF. Suppose the contrary and let A and B be such elements. There is a continuum G' in G which contains A and B and is irreducible with respect to being a continuum and containing A and B. From Lemmas A and B or C with a straightforward argument it follows that if there is any element of G' separating A from ω and not separating B from ω, there is an element A' doing this such that no element of G' separates A' from ω and does not separate B from ω. We note that B is not A' and we denote by A'' the set A if no A' exists or A' if such does exist. Similarly we define a set B'' with respect to A'' and ω. But as G' contains A'' and B'' and is irreducible with

[2] If D and E are two complementary domains of a compact continuous curve M, the outer boundary of D with respect to E is a simple closed curve.

respect to being a continuum and containing A and B, it follows that no element of G' not separated by A'' from B and ω and not separated by B'' from A'' and ω can separate the plane into two complementary domains each of which contains an element of G' such that A'' and B'' are in the same complementary domain. But then as G' is connected there must be uncountably many mutually exclusive open sets in the plane, which is the desired contradiction.

THEOREM II.[3] *If G is connected and every element of G is a simple closed curve, then G is an arc and G^* is an annulus whose boundary simple closed curves are the end elements of G.*

PROOF. From Lemma D we can conclude that of every three elements of G one separates the other two from each other and hence G is an arc. The end elements of G do not separate G and are simple closed curves. One separates the other from ω. Hence they bound an annulus A which contains the other elements of G. From the elementary separation properties of simple closed curves, Lemma A, and the continuity of G, it readily follows that G^* fills up A.

THEOREM III. *If G does not contain any arcs as elements and is connected, then G^* contains a 2-cell.*

PROOF. From Lemma D we conclude again as above that G is an arc. From Lemma B it follows that every cut element of G is a simple closed curve. Hence for a subarc G' of G each element of G' is a simple closed curve and Theorem II implies that G'^* and hence G^* contains a 2-cell.

DEFINITION. A sequence $c_1, c_2, c_3, \cdots$ of simple chains will be said to be hooked provided that c_1 contains an odd number (>7) of links and there exists a point P in the middle link of c_1 and sequences $d_2', d_3', d_4', \cdots$ and $d_2'', d_3'', d_4'', \cdots$ of simple chains such that for each i, (1) c_{i+1} is a closed refinement of c_i and d_{i+1}' and d_{i+1}'' are closed refinements of d_i', (2) d_{i+1}' and d_{i+1}'' are mutually exclusive subchains of c_{i+1}, (3) exactly one link of d_{i+1}' contains P, (4) the end links of d_2' are in the end links of c_1, (5) the end links of d_{i+1}' are in the end links of d_i', (6) d_2'' contains a link in the link of c_1 containing P and a link in one end link of c_1, and (7) d_{i+1}'' contains a link in the link of d_i' containing P and a link in one end link of d_i'.

An ϵ-sequence of chains is a sequence of chains $c_1, c_2, c_3, \cdots$ with the property that if ϵ is any positive number, there exists a number N_ϵ such that for $j>N_\epsilon$, each link of c_j is of diameter less than ϵ.

[3] Theorem II is a special case of Theorem 4 and Lemma 4.3 of reference [6].

LEMMA E. *If M is a continuum which is simply covered by each chain of a hooked ϵ-sequence of chains, then M is not locally connected.*

PROOF. That Lemma E is true follows from the observation that every point P of a continuum M' in the intersection of the d_i' is a limit point of the sum of continua M_i one in each set d_i'' and that every sufficiently small open set containing P has infinitely many components with no two points of distinct M_i in the same component.

THEOREM IV. *If G is connected and contains no simple closed curve as an element, then G^* contains a 2-cell.*

PROOF. Let c_1 be a simple chain of an odd number (>7) of links simply covering an arc t_1 which is an element of G and let d_1 and d_2 be connected open sets one in each of the end links of c_1 such that $\bar{d}_1+\bar{d}_2$ intersects the closure of no cut link of c_1 and each d_i ($i=1, 2$) contains an end point of t_1. From the continuity of the collection it follows that there exists a connected open set E containing t_1 and contained in c_1^* such that every element of G intersecting E lies in E and intersects d_1 and d_2 (and hence every link of c_1). Let G_1 be a subcontinuum of G containing t_1 with G_1^* in E. Let m be the middle link of c_1 and suppose that $m\cdot G_1^*$ contains no 2-cell. Let s_1 and s_2 be arcs which lie in the closures of the end links of c_1 each intersecting the boundary of c_1 exactly in its end points such that for each i ($i=1, 2$) s_i separates d_i from m in the closure of c_1^* and intersects the closure of only one link of c_1 and such that s_1 and s_2 form together with two arcs z_1 and z_2 of the boundary of c_1^* a simple closed curve J whose interior contains all non-end links of c_1 and whose exterior contains d_1+d_2. Let H be a maximal set of mutually exclusive arcs such that each has an end point and only an end point in common with each s_i and each is a subset of an element of G_1. Each element of G_1 contains at least one arc in H. Each arc h of H separates $I=J+\text{Int}\ (J)$ into two mutually separated connected sets and on the basis of this separation property the elements of H form a totally ordered set.

The supposition that $m\cdot G_1^*$ contains no 2-cell implies that there exists a connected open set D in m with the property that H is the sum of two uncountable mutually exclusive sets H_1 and H_2 such that D is separated in I from z_i by every element of H_i. We assume now that no element of G_1 contains two mutually exclusive elements of H. As G_1 is connected and continuous, some point Q must be a limit point of each of the sets H_1^* and H_2^*. Then as follows from sequential limiting properties there exist continua K_1 and K_2 each containing Q

and each intersecting s_1 and s_2, with K_1+K_2 contained in an element f of G_1 and with K_i separating D in I from each element of H_i not in K_i. Let K' be the component of $f \cdot I$ which contains K_1+K_2. K' is locally connected and contains an arc r intersecting each set s_i in exactly an end point of r. $K'+s_1+s_2$ separates D in the plane from all elements of H_1 or H_2 not in K'. Let H_i' denote the set of all elements in H_i not in K'. $H_1'^*$ and $H_2'^*$ are mutually separated and together with the set of elements of H in K', H_1' and H_2' contain all elements of H. By Theorem 41 on p. 216 of reference [2] quoted above the outer boundary with respect to ω of the complementary domain of $K'+s_1+s_2$ containing D is a simple closed curve Y. There exists an arc r' of K' such that r' lies in I, has exactly an end point on each set s_i, and separates $H_1'^*$ from $H_2'^*$ in I, and such that K' contains a point Q in m not in r'. That such r' exists may be seen by considering Y and r. If we let P_1 and P_2 be the points of Y on r closest on r to s_1 and s_2 respectively and note that Y is the sum of two arcs y_1 and y_2 from P_1 to P_2 having just P_1+P_2 in common, at least one of the sets y_1 and y_2 (say y_1) contains a point of m. Let r' be an arc which is contained in $r+y_2$, intersects y_1 in at most P_1 and P_2 and intersects each of the sets s_1 and s_2 in exactly an end point of itself.

As G_1 is connected, f must be a limit element of each of the two subsets of G_1 whose elements contain elements of H_1' and H_2' respectively. But then as the set of arcs of G_1 is dense in G_1, there must be an arc t_2 of G_1 which contains two mutually exclusive subarcs, one in H and the other interesecting m and either s_1 or s_2. We also note that the existence of such an arc t_2 immediately follows if we assume that some element of G_1 contains two elements of H. There exists a chain simply covering t_2 and of diameter less than $1/2$ which satisfies all conditions for the second chain of a hooked sequence of chains whose first chain is c_1 where the point P of the definition could be any point of $m \cdot d_2'$. By an iteration of the above argument without essential modification it can be established that there exists a hooked ϵ-sequence of simple chains c_i each term of which simply covers an element of G. But the common part of the c_i is not locally connected by Lemma E and is locally connected as an element of G by hypothesis. Therefore G^* contains a 2-cell.

THEOREM V. *If G is connected, G^* contains a 2-cell.*

PROOF. Theorem V follows immediately from Theorems I, III, and IV.

THEOREM VI. *If G is connected, then G with respect to its elements as* *points is a hereditary continuous curve such that the closure of the set of emanation points of G is totally disconnected.*

PROOF. Suppose some subcontinuum G' of G is not locally connected at some point P. Then there must be a subcontinuum G'' of G' such that G' is not locally connected at any point of G'' and such that each point of G'' is a limit point of $G'-G''$. By Theorem V, G''^* must contain a 2-cell K. But as the collection G' is continuous and hence as no element of G'' containing an interior point of K can be a limit element of $G'-G''$, we have a contradiction. We therefore conclude that G is a hereditary continuous curve. Suppose now that the closure of the set H of emanation points of G is not totally disconnected and let H' be an arc in $\overline{H}$. But as above H'^* contains a 2-cell and every point in H' is a limit point of $G-H'$ which implies a contradiction similar to that above. Thus Theorem VI is established.

It should be noted that Theorem VI gives necessary but not sufficient conditions which the continuum G must satisfy. Sufficiency conditions would seem to be considerably more complicated. Presumably they would involve among other things use of Theorems I and VI and Lemma C in conjunction with the well known theorem of reference [2] that the continuous monotone image of a sphere is a cactoid. Theorem VII of this paper establishes that if G is restricted to being a subset of a plane, then the conditions of Theorem VI are sufficient in this special case.

We now consider some examples of collections G in which G is an arc and the elements of G are arcs.

EXAMPLE 1. If M is the unit square in the plane, the collection G of vertical line intervals of length 1 in M is a continuous collection of mutually exclusive arcs filling up M.

EXAMPLE 2. If M is the unit square in the plane, there exists a collection G' of mutually exclusive arcs filling up M in which the intervals $[(0, 1/2), (1/4, 1/2)]$ and $[(3/4, 1/2), (1, 1/2)]$ are elements of G' and each other element of G' has both end points on the boundary of M, one with ordinate greater than $1/2$ and the other with ordinate less than $1/2$.

EXAMPLE 3. Let M be the set consisting of the interval I: $[(0, -1), (0, 1)]$, the interval I': $[(2, -1), (2, 1)]$, the curves

$$y_1 = \begin{cases} \sin \dfrac{\pi}{x}, & 0 < x \leqq 1, \\ -\sin \dfrac{\pi}{2-x}, & 1 < x < 2, \end{cases}$$

$$y_2 = \begin{cases} x/10 + \sin \dfrac{\pi}{x}, \ 0 < x \leqq 1, \\ \\ (2 - x)/10 - \sin \dfrac{\pi}{2 - x}, \ 1 < x < 2, \end{cases}$$

and those points on vertical line segments with end points in y_1 and y_2. Let G be a collection filling up M which consists of I, I', and mutually exclusive arcs each of which lies in M and has exactly an end point on y_1 and exactly an end point on y_2 (and hence separates M) with the x-coordinate x_1 of the end point on y_1 and the x-coordinate x_2 of the end point on y_2 satisfying the relationship $x_2 - x_1 = 2x_1x_2$ whenever x_1 and x_2 are both less than $1/2$ and $x_2 - x_1 = 2(2 - x_1)(2 - x_2)$ whenever x_1 and x_2 are both greater than $3/2$. It is clear that such a continuous collection can exist and will be an arc.

EXAMPLE 4. Let M' be the set M of Example 3 plus those points with $0 \leqq x \leqq 2$ and $y_3 < y \leqq 2$ with $y_3(x)$ the maximal ordinate of a point in M for particular x. Let the collection consist of all arcs each of which is an arc of G in Example 3 and a vertical interval from M to the line $y = 2$.

It is not difficult to construct an example which is essentially a modification of Examples 2 and 4 such that the collection G of mutually exclusive arcs is again itself an arc, the set of points in the boundary of M which are accessible from $S - M$ is totally disconnected, and no local cross section of G exists. We also note that if N in the plane is the Cartesian product of a Cantor discontinuum and an interval, j is an integer, and u and v are components of N, there exist j sets of the type of Example 3 with u and v as end elements, each of the j sets intersecting N or any other of the j sets in exactly $u + v$.

THEOREM VII. *If J is a plane hereditary continuous curve such that the closure of the set of emanation points of J is totally disconnected, there exist a continuum M in the plane and a continuous collection G of arcs filling up M such that G with respect to its elements as points is homeomorphic to J.*

PROOF. Let H be the set of all emanation points of J and E the set of all end points of J. $\overline{H} + E$ is a closed and totally disconnected point set, for if P not in $\overline{H}$ is a limit point of E, then it follows directly that J is not locally connected at P and as this observation also implies that E contains at most a countable number of points not in $\overline{H}$, it follows that $\overline{H} + E$ is totally disconnected. $J - (\overline{H} + E)$ is the sum of a countable collection Q of mutually exclusive connected sets open relative to J each a subset of an arc having both end points in $\overline{H}+E$. For any $\epsilon>0$ not more than a finite number of elements of Q are of diameter greater than ϵ (as J is a hereditary continuous curve).

By [3] there exists in the plane an arc t containing $\overline{H}+E$. It is possible that t may intersect J in points other than $\overline{H}+E$ and in fact may have nondegenerate continua in common with J. Let V' be an open curve containing t. There exists an open curve V containing $\overline{H}+E$ such that $V\cdot J$ is totally disconnected. That such a V exists may be easily demonstrated by considering a slight modification of V'. Let $V\cdot J=N$. There exists a homeomorphism of the plane onto itself carrying V onto the x-axis, J onto a set J', and N onto a set N' on the interval $[(0, 0), (1, 0)]$. J' is the sum of N' and a countable collection R of arcs $k_1, k_2, k_3, \cdots$ each having its end points in N' and having no other points on the x-axis. For any $\epsilon>0$ at most a finite number of the elements of R are of diameter greater than ϵ and no point not in N' is common to two elements of R. Let Z be the set of all vertical intervals of length one with midpoints on N'. M will consist of $Z+\sum_{i=1}^{\infty} M_i$ where, for each i, M_i is a set to be defined in terms of $\sum_{j=1}^{i-1} M_j$ and the set k_i.

M_1 is to be a set of the type of Example 3 whose end elements are the elements of Z containing the end points of k_1. M_1 is to contain no points with $y<-1/2$ if k_1 lies (except for its end points) above the x-axis or no points with $y>1/2$ if k_1 lies (except for its end points) below the x-axis. M_1 is to contain no point with $|y|>1/2+$the distance between the end points of k_1, $M_1\cdot Z$ is to be exactly the two elements of Z containing the end points of k_1, and the projection of M_1 on the x-axis is to be the interval whose end points are the end points of k_1. It is clear that such a set M_1 can exist and from the definition of M_1 that there exists a set G_1 of arcs filling up M_1 such that G_1 is an arc whose end elements are in Z. If $\sum_{j=1}^{i-1} M_j$ is defined, then M_i and G_i can be defined similarly from k_i as were M_1 and G_1 from k_1 with the added condition that M_i does not intersect $\sum_{j=1}^{i-1} M_j$. That such an M_i always exists follows in part from the fact that, for $i\neq j$, k_i does not intersect k_j in a cut point of either, and that for any i, k_i lies except for its end points entirely on one side of the x-axis. The set G is the set $Z+\sum_{i=1}^{\infty} G_i$. It is clear from the construction outlined above that M is a compact continuum and that G is homeomorphic with J.

THEOREM[4] VIII. *If M is a 2-cell and G fills up M, then G is an arc.*

PROOF. G must be a dendron, for G is a hereditary continuous curve

[4] G. S. Young has announced a result [5] which implies Theorems VIII and IX.

by Theorem VI and the collection F consisting of the elements of G and the individual points of $S-M$ (where S is a 2-sphere containing M) must by (4) be a cactoid Σ. But the elements of F not in G form an open disc in Σ, and if G contains a simple closed curve it must contain an interior of such simple closed curve in a sphere of Σ and hence G must not contain a simple closed curve. If G contains a branch point g, then g separates G^* into at least three complementary domains and it follows from Lemma B that at least two, say D_1 and D_2, must contain points of the boundary of M. But then some arc t of g with exactly its end points on the boundary of M must separate D_1 and D_2. Then g must be t from the continuity of G and separation properties of arcs and simple closed curves and hence g is not a branch point. Finally, as a dendron without branch points is an arc, Theorem VIII is established.

Theorem IX. *If F is a continuous collection of mutually exclusive locally connected compact continua filling up the plane S and there exist two elements of F which do not separate S, then all elements of F are degenerate.*

Proof. It is clear from the continuity of F that if F contains a nondegenerate element, the set E of all nondegenerate elements of F is open in F. We assume that E exists. If F' is the one point compactification of F then, by [4], F' is a cactoid. If F' contains a 2-sphere U, then some element of F' in U is also in E and as E is open in F' there exists a 2-cell in U each of whose points is an element of E. By Theorem VI this is impossible and hence F' is a dendron. But F' by hypothesis contains at least three elements not separating F' (one the point in F' not in F) and hence F' contains a branch point P, but from Lemma B, P must be a simple closed curve (for it must separate the plane as F fills up the plane) and then from elementary separation properties of simple closed curves P cannot be a branch point of F'. Thus the set E cannot exist under the hypotheses of the theorem and Theorem IX is established.

Theorem X. *If F is a continuous collection of mutually exclusive compact locally connected continua filling up the plane S, then either G is the collection of all points of S or G is a ray the end element of which does not contain a simple closed curve (and may be degenerate) and all other elements of which are simple closed curves.*

Proof. Theorem X can be readily established from Theorem IX and Lemma B.

References

1. R. L. Moore, *Concerning triods in the plane and the junction points of plane continua*, Proc. Nat. Acad. Sci. U.S.A. vol. 14 (1928) pp. 85–88.

2. ———, *Foundations of point set theory*, Amer. Math. Soc. Colloquium Publications, vol. 13, 1932.

3. R. L. Moore and J. R. Kline, *On the most general plane closed point set through which it is possible to pass a simple continuous arc*, Ann. of Math. vol. 20 (1919) pp. 218–223.

4. R. L. Moore, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc. vol. 27 (1925) pp. 416–428.

5. G. S. Young, *Interior and border transformations on surfaces*, Bull. Amer. Math. Soc. Abstract 52-5-220.

6. Herman Cohen, *Some results concerning homogeneous plane continua*, Duke Math. J. vol. 18 (1951) pp. 467–474.

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DIMENSION AND DISCONNECTION

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Let X be a semi-compact separable metric space. We shall prove the following theorem using results found in Hurewicz and Wallman's book *Dimension theory* (Princeton University Press, 1948):

THEOREM. dim $X \leqq n$ *if and only if any closed subset of* X *containing at least two points can be disconnected by a closed set of dimension* $\leqq n-1$.

The necessary and sufficient condition stated in the theorem was found in looking for an n-dimensional analogue of the property of a space being totally disconnected (property α_0 below) and will be denoted by α_n.

Hurewicz and Wallman show (p. 20) that the following three properties of the space X are equivalent:

α_0. X is totally disconnected.

β_0. Any two points in X can be separated.

γ_0. Any point can be separated from a closed set not containing it, that is, dim $X=0$.

They also show (p. 36) that the following n-dimensional analogues of β_0 and γ_0 are equivalent:

β_n. Any two points in X can be separated by a closed set of dimension $\leqq n-1$.

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