APPROXIMATELY CONVEX FUNCTIONS

D. H. HYERS AND S. M. ULAM

In previous papers approximately linear functions [1] and approximately isometric transformations [2; 3; 4] have been studied.[1] In both cases it was shown that the properties of linearity and isometry are "stable" in a certain sense. For example, it was proved that if a function $f(x)$ satisfies the linear functional equation within an amount ϵ, that is, $|f(x+y)-f(x)-f(y)| \leqq \epsilon$, then there exists an actual solution $g(x)$ of the linear functional equation such that $|g(x)-f(x)| \leqq \epsilon$, where ϵ is a given positive number.

In the present paper we discuss a similar problem for the property of convexity. We consider real-valued functions defined on subsets of n-dimensional Euclidean space E_n. A function $f(x)$ defined on a convex subset S of E_n will be called *ϵ-convex* if $f(hx+(1-h)y) \leqq hf(x)+(1-h)\, f(y)+\epsilon$, for all x and y in S and for $0 \leqq h \leqq 1$. Here ϵ is a fixed positive number. Our object is to show that to an ϵ-convex function $f(x)$ there corresponds a convex function $g(x)$ such that $|f(x)-g(x)| \leqq k\epsilon$, for some constant k. In order to prove this we need some results on ϵ-convex functions and on approximating simplices given in the following four lemmas. The paper is self-contained.

LEMMA 1. *Let $f(x)$ be an ϵ-convex function defined on an n-dimensional simplex $S \subset E_n$. Let the vertices of the simplex be $p_0, p_1, \cdots, p_n$, then if $x=\sum_{i=0}^n \alpha_i p_i$, $\alpha_i > 0$, $\sum_{i=0}^n \alpha_i = 1$ is any point of S, we have*

$$f(x) \leqq \sum_{i=0}^{n} \alpha_i f(p_i) + 2k_n\epsilon, \tag{1}$$

where $k_n=(n^2+3n)/(4n+4)$.

PROOF. We prove the inequality by induction on n. For $n=1$, (1) reduces to the statement of ϵ-convexity, so it is true for $n=1$. We assume that (1) holds for n replaced by $n-1$, and prove it for n dimensions. The case in which some $\alpha_i=1$ is trivial, for in this case $x=p_i$, so we may assume that $\alpha_i<1$ for $i=1, \cdots, n+1$. For convenience we may also assume that $\alpha_n \geqq \alpha_j$, $j=0, \cdots, n-1$. Put $h=1-\alpha_n$, $a_j=\alpha_j/h$, $j=0, \cdots, n-1$, and $q=\sum_{j=0}^{n-1} a_j p_j$. Then $x = \sum_{i=0}^n \alpha_i p_i = hq+(1-h)p_n$, and since f is ϵ-convex,

$$f(x) \leqq hf(q) + (1-h)f(p_n) + \epsilon. \tag{2}$$

Presented to the Society, April 28, 1951; received by the editors February 6, 1952.

[1] For a discussion of these and other related questions, see [6].

By the induction hypothesis,

$$f(q) \leqq \sum_{j=0}^{n-1} a_j f(p_j) + \frac{(n-1)(n+2)}{2n}\epsilon. \tag{3}$$

Substituting (3) into (2), we get

$$f(x) \leqq \sum_{i=0}^{n} \alpha_i f(p_i) + \left\{1 + \frac{h(n-1)(n+2)}{2n}\right\}\epsilon. \tag{4}$$

Since $\alpha_n \geqq \alpha_j$, $j=0, \cdots, n-1$, the minimum value which α_n can have is $1/(n+1)$, so the maximum value which h can have is $1-1/(n+1)=n/(n+1)$. Consequently an upper bound for the expression in brackets in inequality (4) is

$$1 + \frac{(n-1)(n+2)}{2(n+1)} = \frac{n^2+3n}{2n+2}\cdot$$

Thus the lemma has been established.

LEMMA 2. *Let $f(x)$ be an ϵ-convex function defined on an open convex set $G\subset E_n$. Then on each closed bounded subset B of G, $f(x)$ is bounded.*

PROOF. f is bounded from above, since B may be covered with a finite number of n-dimensional simplices, each contained in G, and f is bounded on each simplex by Lemma 1.

To prove that f is bounded from below on B, let B be covered with a finite number of closed spheres S_i, such that each S_i is contained in G. Let x_i be the center of S_i, and let x_i+y be any point of the sphere S_i. Then by ϵ-convexity

$$f(x_i) \leqq 2^{-1}f(x_i+y) + 2^{-1}f(x_i-y) + \epsilon,$$

or

$$f(x_i+y) \geqq 2f(x_i) - f(x_i-y) - 2\epsilon.$$

Now x_i-y belongs to the sphere S_i, and since S_i is a closed subset of G, $f(x_i-y)$ is bounded from above as x_i+y varies over S_i. Hence $f(x_i+y)$ is bounded from below for $x_i+y\in S_i$, and it follows that f is bounded from below on B. The proof of the following two lemmas is left to the reader.

LEMMA 3. *Let x lie in an n-dimensional simplex with vertices $q_0, q_1, \cdots, q_n$, so that $x=\sum_{i=0}^{n}\alpha_i q_i$, $\alpha_i\geqq 0$, and $\sum_{i=0}^{n}\alpha_i=1$. Suppose also that we have $n+1$ sequences $\{q_i^{(\nu)}\}$ $(i=0, \cdots, n; \nu=1, 2, 3, \cdots)$ of points such that $q_i^{(\nu)}\to q_i$ as $\nu\to\infty$, and that for each ν, x also lies in the n-dimensional simplex with vertices $q_i^{(\nu)}$ so that*

$$x = \sum_{i=0}^{n} \alpha_i^{(\nu)} q_i^{(\nu)},$$

where $\alpha_i^{(\nu)} \geqq 0$ *and* $\sum_{i=0}^{n} \alpha_i^{(\nu)} = 1$. *Then as* $\nu \to \infty$, $\alpha_i^{(\nu)} \to \alpha_i$.

LEMMA 4. *Suppose* x *is interior to an* n*-dimensional simplex in* E_n *whose vertices are* $q_0, q_1, \cdots, q_n$. *Then if* $q_i^{(\nu)} \to q_i$ *in* E_n *as* $\nu \to \infty$ $(i = 0, \cdots, n)$, x *is also interior to the simplex* $S_n^{(\nu)}$ *whose vertices are* $q_i^{(\nu)}$ $(i = 0, \cdots, n)$, *for sufficiently large* n.

THEOREM 1. *Let* $f(x)$ *be* ϵ*-convex on an open convex set* $G \subset E_n$, *and let* B *be any closed bounded convex subset of* G. *Then there exists a convex function* $\phi(x)$ *on* B *such that*

$$|\phi(x) - f(x)| \leqq k_n \epsilon, \qquad \text{for } x \in B,$$

where $k_n = (n^2 + 3n)/(4n + 4)$.

PROOF. Let H be a bounded convex open subset of G such that $B \subset H$, and $\overline{H} \subset G$. Since B is a compact subset of the open convex set G, the existence of such an H is easily shown. Let K denote the convex hull of the closure of the graph of the function $f(x)$ for $x \in \overline{H}$, so that K is a convex set in E_{n+1}.

Define, for $x = (x_1, \cdots, x_n) \in \overline{H}$, $g(x) = \inf [y; (x_1, x_2, \cdots, x_n, y) \in K]$. Since $f(x)$ is bounded on $\overline{H}$ by Lemma 2, K is a compact set in E_{n+1} and $g(x)$ is well defined on $\overline{H}$. It is easily seen that $g(x)$ is a convex function, and that $g(x) \leqq f(x)$ for $x \in H$. Given a point $x \in B$, let p denote the point $(x_1, x_2, \cdots, x_n, g(x))$ in E_{n+1}. Now p evidently belongs to the boundary of K, and since K is closed, it also belongs to K. By a well known theorem,[2] p lies on an m-dimensional simplex S_m whose vertices are points or limit points of the graph of $f(x)$ for $x \in \overline{H}$, where $m \leqq n+1$. Notice that the assertion is actually true for some $m \leqq n$, for if p were in the interior of an $(n+1)$-dimensional simplex with vertices in K, then p would lie in the interior of K and not on its boundary.

There are three possible cases.

(i) p is a point of the graph of f.

(ii) p is a limit point of the graph of f.

(iii) p is "interior"[3] to some simplex S whose dimension is positive and less than or equal to m, and whose vertices are points or limit points of the graph of f.

In case (i), $f(x) = g(x)$, and there is nothing to prove. In case (ii)

[2] See [5, p. 9].

[3] A point will be called "interior" to a simplex S of dimension r if it belongs to S but not to any face of lower dimension than r.

it is convenient to translate the axes so that the origin of coordinates lies at the point x so that $x=0$. Then by hypothesis there exists a sequence of distinct points $x^{(\mu)}\in H\subset E_n$ tending to zero such that $\lim_{\mu\to\infty} f(x^{(\mu)})=g(0)$. It is clear that an infinite number of these points must all lie in some one of the 2^n-tants determined by the coordinate hyperplanes. For definiteness, let us assume the first 2^n-tant contains an infinite number of these points. We denote them by $x^{(\nu)}$, so that all the coordinates of each $x^{(\nu)}$ may be assumed to be non-negative. Now choose on each coordinate axis a point p_j whose jth coordinate is negative, the others being zero, $j=1, 2, \cdots, n$, such that $p_j\in H$. Consider the simplex $S^{(\nu)}$ whose vertices are $p_1, p_2, \cdots, p_n$ and $x^{(\nu)}$. Then the origin belongs to this simplex, and there exist $\alpha_i^{(\nu)}$, $i=1, \cdots, n+1$, such that

$$\sum_{j=1}^{n} \alpha_j^{(\nu)} p_j + \alpha_{n+1}^{(\nu)} x^{(\nu)} = 0, \tag{5}$$

where $\alpha_j^{(\nu)}\geqq 0$, $\alpha_{n+1}^{(\nu)}>0$, and $\sum_{i=1}^{n+1} \alpha_i^{(\nu)}=1$.

To prove this, let p_{jj} be the jth coordinate of the point p_j and let $x_j^{(\nu)}$ be the jth coordinate of the point $x^{(\nu)}$. Then the "vector" equation (5) may be written in the form:

$$\alpha_j^{(\nu)} p_{jj} + \alpha_{n+1}^{(\nu)} x_j^{(\nu)} = 0, \qquad j = 1, \cdots, n, \tag{6}$$

where $p_{jj}<0$, and $x_j^{(\nu)}\geqq 0$. Since $x^{(\nu)}\neq 0$, at least one of the $x_j^{(\nu)}$ must be positive. If $x_j^{(\nu)}=0$, choose $\alpha_j^{(\nu)}=\rho_j^{(\nu)}=0$. If $x_j^{(\nu)}\neq 0$ equation (6) determines the ratio $\rho_j^{(\nu)}=\alpha_j^{(\nu)}/\alpha_{n+1}^{(\nu)}$, which in this case is evidently positive. The value of $\alpha_{n+1}^{(\nu)}$ is then determined by the requirement that $\sum_{i=1}^{n+1} \alpha_i^{(\nu)}=(1+\sum_{j=1}^{n} \rho_j^{(\nu)})\alpha_{n+1}^{(\nu)}=1$. Thus relation (6) is established. By Lemma 1, it follows that

$$f(0) \leqq \sum_{j=1}^{n} \alpha_j^{(\nu)} f(p_j) + \alpha_{n+1}^{(\nu)} f(x^{(\nu)}) + 2k_n\epsilon. \tag{7}$$

Now as $\nu\to\infty$, $x^{(\nu)}\to 0$. Hence by (6), $\alpha_j^{(\nu)}\to 0$ for $j=1, \cdots, n$. It follows that $\alpha_{n+1}^{(\nu)}\to 1$. Since $f(x^{(\nu)})\to g(0)$, we have $f(0)\leqq g(0)+2k_n\epsilon$, or $f(x)\leqq g(x)+2k_n\epsilon$.

We now turn to case (iii). Here p lies in the interior of an r-dimensional simplex S_r $(1\leqq r\leqq n)$ whose vertices p_i $(i=0, 1, \cdots, r)$ are points or limit points of the graph of f.

Let π be a supporting hyperplane of $K\subset E_{n+1}$ through the point p. Now p is interior to at least one line segment S_1 belonging to S_r and hence to K. Any such line segment S_1 must lie in the hyperplane π, for otherwise S_1 would pierce the hyperplane π at p so that part of

S_1 would lie on one side of π and part on the other, which is impossible since all of K lies on one side of π. It follows that S_r, and hence its vertices p_i, lies in π, and the p_i are boundary points of K.

This supporting hyperplane π cannot be perpendicular to E_n, for in this case π would project (orthogonally) into a hyperplane in E_n which would be a supporting hyperplane of the projection of the convex set K and which would contain the point x. Thus x would be on the boundary of the projection of K. But the projection of K includes the open set H which by hypothesis contains x, so x cannot lie on the boundary of K's projection, and we have a contradiction.

Therefore the projection of S_r onto E_n is a simplex Σ_r of the same dimension r, and the interior of S_r projects into the interior of Σ_r, so that the point x which is the projection of p lies in the interior of Σ_r.

We use double subscripts to denote the coordinates of the vertices p_i of S_r, and we denote the projections of these vertices onto E_n by $q_0, q_1, \cdots, q_r$. Then by hypothesis there exist sequences $q_i^{(\nu)}$ such that $p_{i,n+1} = \lim_{\nu\to\infty} f(q_i^{(\nu)})$, where $\lim_{\nu\to\infty} q_i^{(\nu)} = q_i$, and $q_0, \cdots, q_r$ are the vertices of the r-dimensional simplex $\Sigma_r \subset E_n$, which contains the point x in its interior. Our object is to construct a simplex $S_n^{(\nu)}$ of dimension n in E_n such that x is interior to $S_n^{(\nu)}$, and such that r of its vertices are points $q_0^{(\nu)}, \cdots, q_r^{(\nu)}$. We can then apply Lemma 1 to this simplex and take the limit in the resulting inequality as $\nu\to\infty$.

Suppose first that $r = n$. In this case, x is interior to the n-dimensional simplex $\Sigma_n \subset E_n$, so that

$$x = \sum_{i=0}^{n} \alpha_i q_i, \qquad \alpha_i > 0, \qquad \sum_{i=0}^{n} \alpha_i = 1.$$

Since $q_i^{(\nu)} \to q_i$ in E_n as $\nu\to\infty$, it follows by Lemma 4 that $x = \sum_{i=0}^{n} \alpha_i^{(\nu)} q_i^{(\nu)}$, $\alpha_i^{(\nu)} > 0$, $\sum_{i=0}^{n} \alpha_i^{(\nu)} = 1$. Hence by Lemma 3, $\alpha_i^{(\nu)} \to \alpha_i$ as $\nu\to\infty$.

Now by Lemma 1, we have $f(x) \leqq \sum_{i=0}^{n} \alpha_i^{(\nu)} f(q_i^{(\nu)}) + 2k_n\epsilon$. By taking limits as $\nu\to\infty$ we get

$$f(x) \leqq \sum_{i=0}^{n} \alpha_i p_{i,n+1} + 2k_n\epsilon = g(x) + 2k_n\epsilon.$$

Now let us suppose that $1 \leqq r \leqq n$. Let F_r be the r-dimensional flat containing Σ_r. Now if for all but a finite number of ν's, the $q_i^{(\nu)}$, $i = 0, \cdots, n$; $\nu = 1, 2, 3, \cdots$, are contained in F_r, then $q_i^{(\nu)} \to q_i$ in F_r and one has essentially case (iiia) with r replacing n, so the proof follows as before.

Next suppose that an infinity of points $q_i^{(\nu)}$ for some i lie outside this flat. We may as well assume (by relabeling and suppressing a subsequence if necessary) that all of the $q_0^{(\nu)}$ lie outside F_r.

Let us choose a new coordinate system with origin at q_0 and with the first r axes belonging to F_r, so that the equations of F_r are $z_j=0$, $j=r+1, \cdots, n$. The last $n-r$ coordinates $q_{0,r+1}^{(\nu)}, \cdots, q_{0,n}^{(\nu)}$ of the point $q_0^{(\nu)}$ cannot all be zero for any ν. It follows that for some fixed j, $q_{0,j}^{(\nu)}\neq 0$, for all ν. We may without loss of generality assume that $q_{0,r+1}^{(\nu)}\neq 0$, for all ν. Now there must be an infinity of the numbers $q_{0,r+1}^{(\nu)}$ which are either all positive or all negative, and by reversing the $(r+1)$st coordinate axis if necessary, we may assume that $q_{0,r+1}^{(\nu)}>0$ for all ν.

Next, if $r+1<n$, we consider $q_{0,r+2}^{(\nu)}$. If $q_{0,r+2}^{(\nu)}=0$ for all but a finite number of ν's, we rotate the z_{r+1} and z_{r+2} axes through an acute angle, keeping all of the other axes fixed, in such a way that after the rotation $q_{0,r+1}^{(\nu)}$ will still be positive and $q_{0,r+2}^{(\nu)}$ will become positive for all but a finite number of ν's.

On the other hand if $q_{0,r+2}^{(\nu)}\neq 0$ for an infinite number of ν's, then for an infinite number of ν's, these numbers are all positive or all negative. By reversing the z_{r+2}-axis if necessary we have $q_{0,r+2}^{(\nu)}>0$ for an infinite number of ν's. Thus by suppressing a subsequence if necessary we can arrange matters so that $q_{0,r+1}^{(\nu)}>0$ and $q_{0,r+2}^{(\nu)}>0$ for all ν.

If $r+2<n$, we proceed in the same way, with $r+1$ replacing r, and so on. Thus, there will exist a coordinate system in E_n and sequences of points $q_i^{(\nu)}\to q_i$ $(i=0, 1, \cdots, r)$ such that the origin lies at the point q_0, and $q_{i,j}=0$, $q_{0,j}^{(\nu)}>0$ for $j=r+1, \cdots, n$, where $f(q_i^{(\nu)})\to p_{i,n+1}$, $x=\sum_{i=0}^r \alpha_i q_i$, $g(x)=\sum_{i=0}^r \alpha_i p_{i,n+1}$, $\sum_{i=0}^r \alpha_i=1, \alpha_i>0$.

Now let q_i $(i=r+1, \cdots, n)$ be a point in H whose $(r+1)$st coordinate is a negative number and whose other coordinates are all zero. We now show that x is interior to the n-dimensional simplex whose vertices are $q_0^{(\nu)}, q_1, q_2, \cdots, q_n$, for sufficiently large ν.

Thus we must show the existence of positive numbers β_i $(i=0, \cdots, n)$ with $\sum_{i=0}^n \beta_i=1$ such that $x=\sum_{i=0}^r \alpha_i q_i=\beta_0 q_0^{(\nu)} + \sum_{i=0}^n \beta_i q_i$. That is, the β_i are to satisfy the following system of $n+1$ linear equations:

$$
\begin{aligned}
&\beta_0 q_{0,j}^{(\nu)} + \sum_{i=0}^{r} \beta_i q_{i,j} = \sum_{i=0}^{r} \alpha_i q_{i,j} \qquad (j = 1, \cdots, r),\\
&\beta_0 q_{0,j}^{(\nu)} + \beta_j q_{j,j} = 0 \qquad (j = r+1, \cdots, n),\\
&\sum_{i=0}^{n+1} \beta_i = 1.
\end{aligned}
\tag{8}
$$

Since $\alpha_i>0$, $\sum_{i=0}^{r}\alpha_i=1$, and $q_{0,j}^{(\nu)}\to q_{0,j}=0$, it follows that for $0<\beta_0<1$ there will exist a ν_0, independent of β_0, such that the first r equations of the system (8) have solutions for β_i, $i=1,\cdots,r$, which are between zero and one, whenever $\nu\geqq\nu_0$. Since $q_{j,j}$ and $q_{0,j}^{(\nu)}$ are of opposite signs by construction for $j=r+1,\cdots,n$, it is clear that the next $n-r$ equations will also have solutions β_j, $j=r+1,\cdots,n$, which are between zero and one when β_0 is, and when ν is sufficiently large. With the help of the last equation all the β's may be determined, with $0<\beta_i<1$, $i=0,\cdots,n$.

Next, for a given ν, so large that x is interior to the simplex with vertices $q_0^\nu, q_1,\cdots,q_n$, there will exist by Lemma 4 an index $\mu=\mu(\nu)$ such that x is also interior to the simplex with vertices $q_0^\nu, q_1^\mu, q_2^\mu,\cdots, q_r^\mu, q_{r+1},\cdots,q_n$. Let one such index μ be determined for each ν and put $\bar{q}_i^\nu=q_i^{\mu(\nu)}$, $i=1,\cdots,r$. For convenience we also put $\bar{q}_0^\nu=q_0^\nu$. Then there exist $\alpha_i>0$, $i=0,1,\cdots,n$, such that $\sum_{i=0}^{n}\alpha_i^\nu=1$ and $x=\sum_{i=0}^{r}\alpha_i q_i=\sum_{i=0}^{r}\alpha_i^\nu\bar{q}_i+\sum_{i=r+1}^{n}\alpha_i q_i$. By Lemma 1 we have

$$f(x)\leqq\sum_{i=0}^{r}\alpha_i f(\bar{q}_i^\nu)+\sum_{j=r+1}^{n}\alpha_j f(q_j)+2k_n\epsilon.$$

Now as $\nu\to\infty$, $\bar{q}_i^\nu\to q_i$, $f(\bar{q}_i^\nu)\to p_{i,n+1}$, and, by Lemma 3, we know that $\alpha_i^\nu\to\alpha_i$ for $i=0,1,\cdots,r$, while $\alpha_j^\nu\to 0$, $j=r+1,\cdots,n$. Hence by letting $\nu\to\infty$ in the last inequality we get

$$f(x)\leqq\sum_{i=0}^{r}\alpha_i p_{i,n+1}+2k_n\epsilon=g(x)+2k_n\epsilon.$$

We have proved that for any point $x\in B$, $g(x)\leqq f(x)\leqq g(x)+2k_n\epsilon$, where $g(x)$ is a convex function. Now define $\phi(x)=g(x)+k_n\epsilon$. Then $\phi(x)$ is convex and

$$|\phi(x)-f(x)|\leqq k_n\epsilon \qquad \text{for } x\in B.$$

This completes the proof of theorem 1.

THEOREM 2. *If $f(x)$ is an ϵ-convex function defined on a convex open subset of G of E_n, then there exists a convex function $\phi(x)$ defined on G such that* $|f(x)-\phi(x)|\leqq k_n\epsilon$.

PROOF. Let H_ν, $\nu=1,2,3,\cdots$, be a sequence of convex, compact subsets of G such that $H_{\nu+1}\subset H_\nu$ and such that $G=\bigcup_{\nu=1}^{\infty}H_\nu$ (the existence of such a sequence is easily demonstrated). Then by Theorem 1, there exists for each ν a convex function $\phi_\nu(x)$ on H_ν such that $|\phi_\nu(x)-f(x)|\leqq k_n\epsilon$, for $x\in H_\nu$. For each fixed positive integer μ, the function $f(x)$ is bounded on H_μ by Lemma 2. Hence the sequence $\{\phi_\nu(x)\}$ is defined and uniformly bounded on H_μ for $\nu\geqq\mu$. By a well known selection theorem there exists a subsequence $\{\phi_{1p}(x)\}$ of the $\phi_\nu(x)$ which converges for all $x \in H_1$. Similarly there is a subsequence $\{\phi_{2p}(x)\}$ of the $\phi_{1p}(x)$ which is defined and convergent on H_2, and so on. Now consider the sequence $\{\phi_{pp}(x)\}$, $p=1, 2, 3, \cdots$. For any given $x \in G$, there exists a positive integer m so that $x \in H_m$. Hence for $p \geqq m$, the sequence $\{\phi_{pp}(x)\}$ is defined and converges to a limit $\phi(x)$. Thus $g(x)$ is defined, is convex, and satisfies the inequality $|\phi(x)-f(x)| \leqq k_n\epsilon$ for $x \in G$.

References

1. D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. vol. 27 (1941) pp. 222–224.

2. D. H. Hyers and S. M. Ulam, *On approximate isometries*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 288–292.

3. ———, *Approximate isometries of the space of continuous functions*, Ann. of Math. vol. 48 (1947) pp. 285–289.

4. D. G. Bourgin, *Approximate isometries*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 704–714.

5. T. Bonnesen and W. Fenchel, *Konvexe Körper*, New York, 1948.

6. D. G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc. vol. 57 (1951) pp. 223–237.

The University of Southern California and
The Los Alamos Scientific Laboratory