A CLASS OF MULTIVALENT FUNCTIONS WITH ASSIGNED ZEROS

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1. **Introduction.** Recently A. W. Goodman [1; 2] has studied the following two classes of multivalent functions:

(i) p-valently starlike functions denoted by $S(p)$: A function $f(z)$ is said to be p-valently starlike with respect to the origin for $|z|<1$ if (a) $f(z)$ is regular and p-valent for $|z|<1$ and (b) if there exists a ρ such that, for each r in $\rho<r<1$, the radius vector joining the origin to $f(re^{i\theta})$ turns continuously in the counterclockwise direction and makes p complete revolutions as θ varies from 0 to 2π.

(ii) Typically-real functions of order p denoted by $T(p)$. A function

$$f(z) = \sum_{n=0}^{\infty} b_n z^n \tag{1.1}$$

is said to be typically-real of order p if in (1.1) the coefficients b_n are all real and if $f(z)$ is regular in $|z| \leqq 1$ and $\Im f(e^{i\theta})$ changes sign $2p$ times as θ traverses the boundary of the unit circle.

Concerning the above classes of functions he obtained the following results:

Let

$$f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n \tag{1.2}$$

be a function of the set $S(p)$ or $T(p)$. Suppose that in addition to the qth order zero at $z=0$, the function $f(z)$ has exactly $p-q$ zeros, $\beta_1, \beta_2, \cdots, \beta_{p-q}$, such that $0<|\beta_j|<1, j=1, 2, \cdots, p-q$. Then

$$|a_n| \leqq A_n, \qquad n = q+1, q+2, \cdots \tag{1.3}$$

where A_n is defined by

$$\begin{aligned} F(z) &= \frac{z^q}{(1-z)^{2p}} \prod_{j=1}^{p-q}\left(1+\frac{z}{|\beta_j|}\right)(1+z|\beta_j|) \\ &= z^q + \sum_{n=q+1}^{\infty} A_n z^n. \end{aligned}$$

The inequality (1.3) is sharp.

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For functions of the set $T(p)$ he has obtained a more general result [2; 3]. However even that result cannot include the above result for $S(p)$ since in $S(p)$ the coefficients can be complex.

Now in the present paper we shall introduce a wider class of functions $D(p)$ which includes $S(p)$, $T(p)$ and others in the case where $f(z)$ has p zeros, proving that the inequality (1.3) is also valid for the functions of this class.

2. Preliminary considerations.

LEMMA 1. *Let*

$$w = f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2.1}$$

be regular for $|z| \leqq 1$ *and have* p $(\geqq 0)$ *zeros in* $|z| \leqq 1$. *Then there exists a point* ζ $(|\zeta| = 1)$ *for which the following equality holds*

$$\arg f(-\zeta) = \arg f(\zeta) + p\pi. \tag{2.2}$$

PROOF. Without loss of generality, let $\arg f(-1) - \arg f(+1) < p\pi$. If a point ζ moves from $+1$ to -1, $\arg f(-\zeta) - \arg f(\zeta)$ varies continuously from $\arg f(-1) - \arg f(+1) < p\pi$ to $2p\pi - (\arg f(-1) - \arg f(+1)) > p\pi$, since $f(z)$ has p zeros. Hence at a point ζ the equality (2.2) holds.

The special cases of Lemma 1 and the following Definition 1 we owe to N. G. DeBruijn [4] and S. Ozaki [5].

DEFINITION 1. Let us say the diametral line of $f(z)$ for the straight line $[f(\zeta)0f(-\zeta)]$ when ζ satisfies Lemma 1.

Accordingly we have the following:

LEMMA 1′. *Let* (2.1) *be a function regular for* $|z| \leqq 1$. *Then there exists at least one diametral line of* $f(z)$ *in the* w*-plane.*

DEFINITION 2. Let $f(z)$ be regular for $|z| \leqq 1$ and let C be the image curve of $|z| = 1$. If C is cut by a straight line passing through the origin in $2p$, and not more than $2p$ points, then $f(z)$ is said to be starlike of order p in the direction of the straight line. Especially when the direction of starlikeness of order p is that of the diametral line of $f(z)$, $f(z)$ is said to belong to the class $D(p)$.

The idea of being starlike in one direction was introduced by M. S. Robertson [6] and also extended to general p by him [7; 8]. And $D(1)$ was studied in [4; 5].

LEMMA 2. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a member of the class* $D(p)$. *Further let* $f(z)$ *have* s *zeros* $\beta_1, \beta_2, \cdots, \beta_s$ *such that* $0 < |\beta_j| < 1, j = 1, 2, \cdots, s$.

Then the function $F(z)$ *defined by*

$$F(z) = f(z)g(z), \qquad g(z) = z^s \Big/ \prod_{i=1}^{s} (z - \beta_i)(1 - \bar{\beta}_i z)$$

is also a member of the class $D(p)$.

PROOF. Regularity of $F(z)$ in $|z| \leqq 1$ is evident. Now we easily see that

$$g(e^{i\theta}) = 1 \Big/ \prod_{j=1}^{s} | e^{i\theta} - \beta_j |^2.$$

Hence $\arg F(e^{i\theta}) = \arg f(e^{i\theta})$ for every θ. Consequently if $f(z) \in D(p)$, then $F(z) \in D(p)$.

3. **The main theorem.**

THEOREM 1. *Let*

$$f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n \tag{3.1}$$

be a function of the set $D(p)$. *Suppose that in addition to the* q*th order zero at* $z=0$, *the function* $f(z)$ *has exactly* $p-q$ *zeros*, $\beta_1, \beta_2, \cdots, \beta_{p-q}$, *such that* $0 < |\beta_j| < 1, j = 1, 2, \cdots, p-q$. *Then*

$$| a_n | \leqq B_n, \qquad n = q+1, q+2, \cdots, \tag{3.2}$$

$$| f(re^{i\theta}) | \leqq F(r) \qquad \textit{for } r < 1, \tag{3.3}$$

where B_n *and* $F(r)$ *are defined by*

$$\begin{aligned} F(z) &= \frac{z^q}{(1-z)^{2p}} \prod_{j=1}^{p-q} \left(1 + \frac{z}{|\beta_j|}\right)(1 + z|\beta_j|) \\ &= z^q + \sum_{n=q+1}^{\infty} B_n z^n. \end{aligned} \tag{3.4}$$

PROOF. Let us put

$$E(z) = f(z) \cdot z^{p-q} \Big/ \prod_{i=1}^{p-q} (z - \beta_i)(1 - \bar{\beta}_i z). \tag{3.5}$$

Then by Lemma 2, $E(z) \in D(p)$ since $f(z) \in D(p)$, and

$$\begin{aligned} (-1)^{p-q} \prod_{i=1}^{s} \beta_i E(z) &= z^p + \alpha_{p+1} z^{p+1} + \cdots \\ &= \psi(z) \in D(p). \end{aligned} \tag{3.6}$$

We wish now to show that

$$\psi(z) \ll z^p/(1-z)^{2p}.$$

For the purpose it will be sufficient to assume that the diametral line in whose direction $\psi(z)$ is starlike of order p is $\psi(1)\, 0\, \psi(-1)$, since in the other cases we may consider $\psi(\zeta z) = g(z)$ for which $g(1)\, 0\, g(-1)$ is the diametral line.

Let $\psi(1) = \omega = |\omega| e^{-i\alpha}$; then by our hypothesis

$$\begin{aligned} &\Im e^{i\alpha}\psi(e^{i\theta}) > 0 && \text{for } \theta_{2s-1} < \theta < \theta_{2s}, \\ (3.7) \qquad &\Im e^{i\alpha}\psi(e^{i\theta}) < 0 && \text{for } \theta_{2s} < \theta < \theta_{2s+1}, \end{aligned}$$

$$s = 1, 2, \cdots, p,\ \theta_{2p+1} = \theta_1 + 2\pi,\ \theta_1 = 0,\ \theta_j = \pi,\ 1 < j \leqq 2p.$$

Let

$$(3.8) \qquad \phi(z) = (-1)^{p-1} \exp\left(-\frac{i}{2}\sum_{s=1}^{2p}\theta_s\right)\cdot \prod_{s=1}^{2p}(e^{i\theta_s} - z)/z^p,$$

then

$$(3.9) \qquad \phi(e^{i\theta}) = -2^{2p}\prod_{s=1}^{2p}\sin\frac{\theta_s - \theta}{2}\cdot$$

Hence we obtain

$$\begin{aligned} (3.10) \qquad &\phi(e^{i\theta}) > 0 \quad \text{for} \quad \theta_{2s-1} < \theta < \theta_{2s}, \\ &\phi(e^{i\theta}) < 0 \quad \text{for} \quad \theta_{2s} < \theta < \theta_{2s+1}, \qquad s = 1, 2, \cdots, p. \end{aligned}$$

Let

$$(3.11) \qquad G(z) = -ie^{i\alpha}\psi(z)\phi(z) = e^{i\beta} + \sum_{n=1}^{\infty}\gamma_n z^n,$$

then $G(z)$ is regular for $|z| \leqq 1$ and

$$\Re G(e^{i\theta}) \geqq 0.$$

Accordingly by the principle of minimum for regular harmonic functions

$$\Re G(z) > 0 \qquad \text{for } |z| < 1.$$

Hence by Carathéodory's theorem

$$|\gamma_n| \leqq 2\Re e^{i\beta} \leqq 2 \qquad \text{for } n = 1, 2, \cdots.$$

Consequently

$$(3.12) \qquad G(z) \ll (1+z)/(1-z).$$

On the other hand from (3.11) we have

$$\psi(z) = ie^{-i\alpha}(-1)^p \exp\left(\frac{i}{2}\sum_{s=1}^{2p}\theta_s\right) \cdot z^p G(z) \Big/ \left\{(1-z^2)\prod_{s\neq 1,j}^{2p}(e^{i\theta_s}-z)\right\}$$

which is dominated by

$$z^p\left(\frac{1+z}{1-z}\right)\cdot\frac{1}{1-z^2}\cdot\frac{1}{(1-z)^{2p-2}} = \frac{z^p}{(1-z)^{2p}} \tag{3.13}$$

since we have (3.12).

From (3.4) and (3.5), we have

$$f(z) = \psi(z)\prod_{i=1}^{p-q}(z-\beta_i)(1-\bar{\beta}_i z)\Big/\left(\prod_{i=1}^{p-q}\beta_i z^{p-q}\right)$$

which is dominated by

$$\frac{z^p}{(1-z)^{2p}}\prod_{i=1}^{p-q}\left(1+\frac{z}{|\beta_i|}\right)(1+|\beta_i|z)\cdot\frac{1}{z^{p-q}} = F(z)$$

since we have (3.13). Hence we obtain

$$|a_n| \leqq B_n, \qquad n = q+1, q+2, \cdots,$$

and

$$|f(re^{i\theta})| \leqq F(r) \qquad \text{for } r < 1. \text{ q.e.d.}$$

4. A class of functions related to $D(p)$.

Definition 3. Let $w=f(z)$ be regular for $|z| \leqq 1$ and C be the image curve of $|z| = 1$. Let, further, P be the orthogonal projection of $f(e^{i\theta})$ onto a straight line. Then P will move on the straight line both positively or negatively when θ varies from 0 to 2π. If P changes its direction of movement $2p$ times when θ varies from 0 to 2π, then $f(z)$ is said to be convex of order p in the direction which is perpendicular to the straight line. This class of functions has recently been studied by M. S. Robertson [9].

Especially if, when we represent $f(z)$, $zf'(z)$ in the same plane, the straight line is parallel to a diametral line of $zf'(z)$, then $f(z)$ is said to be a member of $F(p)$.

Lemma 3. *$f(z)$ is a member of the class $F(p)$ if and only if $zf'(z)$ belongs to the class $D(p)$.*

PROOF. This is a generalization of M. S. Robertson's lemma [6].

It is sufficient to prove the lemma in the case where the diametral line of $f(z)$ is the real axis, since in the other cases we may consider $e^{i\alpha}f(z)$ with a suitable choice for the real parameter α.

Using the identity

$$\Im\{zf'(z)\} = -\partial\Re f(z)/\partial\theta \qquad \text{for } |z| = 1$$

we see, under the hypothesis,

$$\Im\{zf'(z)\} = -\partial\Re f(e^{i\theta})/\partial\theta > 0 \qquad \text{for } \theta_{2s-1} < \theta < \theta_{2s},$$
$$\Im\{zf'(z)\} = -\partial\Re f(e^{i\theta})/\partial\theta < 0 \qquad \text{for } \theta_{2s} < \theta < \theta_{2s+1},$$
$$s = 1, 2, \cdots, p,\ \theta_j = \theta_1 + \pi,\ \theta_{2p+1} = \theta_1 + 2\pi.$$

Hence $f(z) \in F(p)$ if and only if $zf'(z) \in D(p)$.

THEOREM 2. *Let*

$$f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n \tag{4.1}$$

be a function of the set $F(p)$. Suppose that in addition to the $(q-1)$th order critical points at $z=0$, the function $f(z)$ has exactly $p-q$ critical points $\alpha_1, \alpha_2, \cdots, \alpha_{p-q}$ such that $0<|\alpha_j|<1, j=1, 2, \cdots, p-q$. Then

$$|a_n| \leqq qC_n/n, \qquad n = q+1, q+2, \cdots, \tag{4.2}$$

$$|f(re^{i\theta})| \leqq q\int_0^r \frac{F(r)}{r}\,dr, \qquad \textit{for } r < 1, \tag{4.3}$$

$$|f'(re^{i\theta})| \leqq qF(r)/r, \qquad \textit{for } r < 1, \tag{4.4}$$

where C_n and $F(r)$ are defined by

$$\begin{aligned} F(z) &= \frac{z^q}{(1-z)^{2p}} \prod_{j=1}^{p-q}\left(1 + \frac{z}{|\beta_j|}\right)(1 + z|\beta_j|) \\ &= z^q + \sum_{n=q+1}^{\infty} C_n z^n. \end{aligned} \tag{4.5}$$

PROOF. Since $f(z) \in F(p)$,

$$\frac{1}{q} zf'(z) = z^q + \frac{1}{q}\sum_{n=q+1}^{\infty} na_n z^n \in D(p)$$

by Lemma 3.

By using the main theorem we have (4.2) and (4.4). By integrating $f'(z)$ along a radius we have, for $z=re^{i\theta}$,

$$|f(re^{i\theta})| = \left| \int_0^z f'(z)dz \right| \leqq \int_0^r |f'(re^{i\theta})| \, dr \leqq q \int_0^r \frac{F(r)}{r} dr$$

for $r < 1$,

which completes the proof.

5. **Subclasses of $D(p)$ and $F(p)$.**

Corollary 1. *Let $f(z)$ in the form* (3.1) *be regular for $|z| \leqq 1$ and assigned with the same zeros as in Theorem* 1. *Suppose that $f(z)$ satisfies one of the following conditions:*

(i) $\Re[zf'(z)/f(z)] > 0$ *for* $|z| = 1$,

(ii) $f(1) =$ *real*, $f(-1) =$ *real and* $\Im f(e^{i\theta})$ *changes sign* $2p$ *times on* $|z| = 1$,

(iii) $f(z) \in T(p)$.

Then (3.2) *and* (3.3) *hold.*

Proof. (i) Since there exists at least one diametral line of $f(z)$ by Lemma 1′, and since $f(z)$ is starlike of order p in every direction by the fact that $\Re[zf'(z)/f(z)] > 0$ on $|z| = 1$ and $f(z)$ has p zeros in $|z| < 1$, $f(z)$ is evidently starlike of order p in the direction of the above diametral line.

(ii) In this case the diametral line of $f(z)$ is evidently the real axis and is starlike of order p in this direction by our hypothesis, which proves the corollary by using the main theorem.

(iii) This is a direct consequence of the preceding (ii).

Corollary 2. *Let $f(z)$ in the form* (4.1) *be regular for $|z| \leqq 1$ and assigned with the same critical points as in Theorem* 2. *Suppose that $f(z)$ satisfies one of the following conditions:*

(i) $1 + \Re[zf''(z)/f'(z)] > 0$ *for* $|z| = 1$.

(ii) $f'(1) =$ *real*, $f'(-1) =$ *real, and $f(z)$ is convex of order p in the direction of the imaginary axis.*

(iii) *In* (4.1) *the coefficients are all real and $f(z)$ is convex of order p in the direction of the imaginary axis.*

Then (4.2), (4.3), *and* (4.4) *hold.*

Proof. (i) By our hypothesis $zf'(z)$ has p zeros in $|z| < 1$ and $\Re[z\{zf'(z)\}'/\{zf'(z)\}] > 0$ on $|z| = 1$. Hence $zf'(z)$ is starlike of order p in every direction. Consequently $zf'(z) \in D(p)$ by Corollary 1 adopting (i). Accordingly $f(z) \in F(p)$ by Lemma 3.

(ii) By our hypothesis $-\partial \Re f(z)/\partial\theta$ changes sign $2p$ times on $|z| = 1$. Accordingly $\Im\{zf'(z)\}$ changes sign $2p$ times on $|z| = 1$ by

Lemma 3. And $1f'(1)=$real, $(-1)f'(-1)=$real. Hence $zf'(z)\in D(p)$. Consequently $f(z)\in F(p)$.

(iii) This is a special case of (ii).

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