A CLASS OF MULTIVALENT FUNCTIONS
WITH ASSIGNED ZEROS

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1. Introduction. Recently A. W. Goodman [1; 2] has studied the
following two classes of multivalent functions:

(i) \( p \)-valently starlike functions denoted by \( S(p) \): A function \( f(z) \) is said to be \( p \)-valently starlike with respect to the origin for \( |z| < 1 \) if (a) \( f(z) \) is regular and \( p \)-valent for \( |z| < 1 \) and (b) if there exists a \( p \) such that, for each \( r \) in \( \rho < r < 1 \), the radius vector joining the origin to \( f(re^{i\theta}) \) turns continuously in the counterclockwise direction and makes \( p \) complete revolutions as \( \theta \) varies from 0 to 2\( \pi \).

(ii) Typically-real functions of order \( p \) denoted by \( T(p) \). A function

\[
f(z) = \sum_{n=0}^{\infty} b_n z^n
\]

is said to be typically-real of order \( p \) if in (1.1) the coefficients \( b_n \)
are all real and if \( f(z) \) is regular in \( |z| \leq 1 \) and \( \Im(f(e^{i\theta})) \) changes sign \( 2p \) times as \( \theta \) traverses the boundary of the unit circle.

Concerning the above classes of functions he obtained the following
results:

Let

\[
f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n
\]

be a function of the set \( S(p) \) or \( T(p) \). Suppose that in addition to the
\( q \)th order zero at \( z = 0 \), the function \( f(z) \) has exactly \( p - q \) zeros,
\( \beta_1, \beta_2, \cdots, \beta_{p-q} \), such that \( 0 < |\beta_j| < 1, j = 1, 2, \cdots, p-q \). Then

\[
|a_n| \leq A_n, \quad n = q + 1, q + 2, \cdots
\]

where \( A_n \) is defined by

\[
F(z) = \frac{z^q}{(1 - z)^{2p}} \prod_{j=1}^{p-q} \left( 1 + \frac{z}{|\beta_j|} \right) (1 + z |\beta_j|)
= z^q + \sum_{n=q+1}^{\infty} A_n z^n.
\]

The inequality (1.3) is sharp.

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For functions of the set $T(p)$ he has obtained a more general result [2; 3]. However even that result cannot include the above result for $S(p)$ since in $S(p)$ the coefficients can be complex.

Now in the present paper we shall introduce a wider class of functions $D(p)$ which includes $S(p)$, $T(p)$ and others in the case where $f(z)$ has $p$ zeros, proving that the inequality (1.3) is also valid for the functions of this class.

2. Preliminary considerations.

**Lemma 1.** Let

$$w = f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be regular for $|z| \leq 1$ and have $p$ ($\geq 0$) zeros in $|z| \leq 1$. Then there exists a point $\xi$ ($|\xi| = 1$) for which the following equality holds

$$\arg f(-\xi) = \arg f(\xi) + \pi \tau.$$  \hspace{1cm} (2.2)

**Proof.** Without loss of generality, let $\arg f(-1) - \arg f(+1) < \pi \tau$. If a point $\xi$ moves from $+1$ to $-1$, $\arg f(-\xi) - \arg f(\xi)$ varies continuously from $\arg f(-1) - \arg f(+1) < \pi \tau$ to $2\pi \tau - (\arg f(-1) - \arg f(+1)) > \pi \tau$, since $f(z)$ has $p$ zeros. Hence at a point $\xi$ the equality (2.2) holds.

The special cases of Lemma 1 and the following Definition 1 we owe to N. G. DeBruijn [4] and S. Ozaki [5].

**Definition 1.** Let us say the diametral line of $f(z)$ for the straight line $[f(\xi)0f(-\xi)]$ when $\xi$ satisfies Lemma 1.

Accordingly we have the following:

**Lemma 1'.** Let (2.1) be a function regular for $|z| \leq 1$. Then there exists at least one diametral line of $f(z)$ in the $w$-plane.

**Definition 2.** Let $f(z)$ be regular for $|z| \leq 1$ and let $C$ be the image curve of $|z| = 1$. If $C$ is cut by a straight line passing through the origin in $2p$, and not more than $2p$ points, then $f(z)$ is said to be starlike of order $p$ in the direction of the straight line. Especially when the direction of starlikeness of order $p$ is that of the diametral line of $f(z)$, $f(z)$ is said to belong to the class $D(p)$.

The idea of being starlike in one direction was introduced by M. S. Robertson [6] and also extended to general $p$ by him [7; 8]. And $D(1)$ was studied in [4; 5].

**Lemma 2.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a member of the class $D(p)$. Further let $f(z)$ have $s$ zeros $\beta_1, \beta_2, \ldots, \beta_s$ such that $0 < |\beta_j| < 1$, $j = 1, 2, \ldots, s$. 
Then the function $F(z)$ defined by
\[ F(z) = f(z)g(z), \quad g(z) = z^p \prod_{i=1}^{p} \frac{1}{(z - \beta_i)(1 - \bar{\beta}_i z)} \]
is also a member of the class $D(p)$.

**Proof.** Regularity of $F(z)$ in $|z| \leq 1$ is evident. Now we easily see that
\[ g(e^{i\theta}) = 1/\prod_{i=1}^{p} |e^{i\theta} - \beta_i|^2. \]
Hence $\arg F(e^{i\theta}) = \arg f(e^{i\theta})$ for every $\theta$. Consequently if $f(z) \in D(p)$, then $F(z) \in D(p)$.

3. The main theorem.

**Theorem 1.** Let
\[ f(z) = z^p + \sum_{n=q+1}^{\infty} a_n z^n \]
be a function of the set $D(p)$. Suppose that in addition to the $q$th order zero at $z=0$, the function $f(z)$ has exactly $p-q$ zeros, $\beta_1, \beta_2, \cdots, \beta_{p-q}$, such that $0 < |\beta_j| < 1$, $j=1, 2, \cdots, p-q$. Then
\[ |a_n| \leq B_n, \quad n = q + 1, q + 2, \cdots, \]
\[ |f(re^{i\theta})| \leq F(r) \quad \text{for } r < 1, \]
where $B_n$ and $F(r)$ are defined by
\[ F(z) = \frac{z^q}{(1 - z)^{p-q}} \prod_{i=1}^{p-q} \left(1 + \frac{z}{|\beta_i|} \right) \left(1 + z|\beta_i|\right) \]
\[ = z^q + \sum_{n=q+1}^{\infty} B_n z^n. \]

**Proof.** Let us put
\[ E(z) = f(z) \cdot z^{p-q} \prod_{i=1}^{p-q} (z - \beta_i)(1 - \bar{\beta}_i z). \]
Then by Lemma 2, $E(z) \in D(p)$ since $f(z) \in D(p)$, and
\[ (-1)^{p-q} \prod_{i=1}^{p-q} \beta_i E(z) = z^p + \alpha_{p+1} z^{p+1} + \cdots \]
\[ = \psi(z) \in D(p). \]
We wish now to show that
\[ \psi(z) \ll z^p/(1 - z)^{2p}. \]

For the purpose it will be sufficient to assume that the diametral line in whose direction \( \psi(z) \) is starlike of order \( p \) is \( \psi(1) \theta \psi(-1) \), since in the other cases we may consider \( \psi'(z) = g(z) \) for which \( g(1) \neq g(-1) \) is the diametral line.

Let \( \psi(1) = \omega = |\omega| e^{-ia\pi} \); then by our hypothesis
\[
\begin{align*}
\Im e^{ia\psi(\omega)} &> 0 & \text{for } \theta_{2s-1} < \theta < \theta_{2s}, \\
\Im e^{ia\psi(\omega)} &< 0 & \text{for } \theta_{2s} < \theta < \theta_{2s+1},
\end{align*}
\]
(3.7) \( s = 1, 2, \cdots, p, \theta_{2p+1} = \theta_1 + 2\pi, \theta_1 = 0, \theta_j = \pi, 1 < j \leq 2p. \)

Let
\[
(3.8) \quad \phi(\omega) = (-1)^{p-1} \exp \left( -\frac{i}{2} \sum_{s=1}^{2p} \theta_s \right) \prod_{s=1}^{2p} \frac{(e^{i\omega} - z)/z^p,}
\]
then
\[
(3.9) \quad \phi(\omega) = -2^{2p} \prod_{s=1}^{2p} \sin \frac{\theta_s - \theta}{2}.
\]

Hence we obtain
\[
\begin{align*}
\phi(\omega) &> 0 & \text{for } \theta_{2s-1} < \theta < \theta_{2s}, \\
\phi(\omega) &< 0 & \text{for } \theta_{2s} < \theta < \theta_{2s+1},
\end{align*}
\]
(3.10) \( s = 1, 2, \cdots, p. \)

Let
\[
(3.11) \quad G(z) = -ie^{i\omega}\psi(z)\phi(z) = e^{i\omega} + \sum_{n=1}^{\infty} \gamma_n z^n,
\]
then \( G(z) \) is regular for \( |z| \leq 1 \) and
\[ \Re G(\omega) \geq 0. \]

Accordingly by the principle of minimum for regular harmonic functions
\[ \Re G(z) > 0 \quad \text{for } |z| < 1. \]

Hence by Carathéodory's theorem
\[ |\gamma_n| \leq 2\Re e^{i\omega} \leq 2 \quad \text{for } n = 1, 2, \cdots. \]

Consequently
\[
(3.12) \quad G(z) \ll (1 + z)/(1 - z).
\]
On the other hand from (3.11) we have
\[ \psi(z) = i e^{-i\alpha}(-1)^p \exp \left( \frac{i}{2} \sum_{s=1}^{2p} \theta_s \right) \]
\[ \cdot z^p G(z) \Bigg/ \left\{ (1 - z^2) \prod_{s=1,s \neq 1}^{2p} (e^{i\theta_s} - z) \right\} \]
which is dominated by
\[ \frac{z^p}{1 + 2z} \cdot \frac{1}{1 - z^2} \cdot \frac{1}{(1 - z)^{2p-2}} = \frac{z^p}{(1 - z)^{2p}} \]
since we have (3.12).

From (3.4) and (3.5), we have
\[ f(z) = \psi(z) \prod_{i=1}^{p-q} (z - \beta_i)(1 - \bar{\beta}_i z) \Bigg/ \left( \prod_{i=1}^{p-q} \beta_i z^{p-q} \right) \]
which is dominated by
\[ \frac{z^p}{(1 - z)^{2p}} \prod_{i=1}^{p-q} \left( 1 + \frac{z}{|\beta_i|} \right)(1 + |\beta_i| \cdot \frac{1}{z^{p-q}}) = F(z) \]
since we have (3.13). Hence we obtain
\[ |a_n| \leq B_n, \quad n = q + 1, q + 2, \ldots, \]
and
\[ |f(re^{i\theta})| \leq F(r) \quad \text{for } r < 1. \text{ q.e.d.} \]

4. A class of functions related to $D(p)$.

**Definition 3.** Let $w = f(z)$ be regular for $|z| \leq 1$ and $C$ be the image curve of $|z| = 1$. Let, further, $P$ be the orthogonal projection of $f(e^{i\theta})$ onto a straight line. Then $P$ will move on the straight line both positively or negatively when $\theta$ varies from 0 to $2\pi$. If $P$ changes its direction of movement $2p$ times when $\theta$ varies from 0 to $2\pi$, then $f(z)$ is said to be convex of order $p$ in the direction which is perpendicular to the straight line. This class of functions has recently been studied by M. S. Robertson [9].

Especially if, when we represent $f(z)$, $zf'(z)$ in the same plane, the straight line is parallel to a diametral line of $zf'(z)$, then $f(z)$ is said to be a member of $F(p)$.

**Lemma 3.** $f(z)$ is a member of the class $F(p)$ if and only if $zf'(z)$ belongs to the class $D(p)$. 

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Proof. This is a generalization of M. S. Robertson's lemma [6].

It is sufficient to prove the lemma in the case where the diametral line of \( f(z) \) is the real axis, since in the other cases we may consider \( e^{i\alpha}f(z) \) with a suitable choice for the real parameter \( \alpha \).

Using the identity

\[
\Im \{zf'(z)\} = -\frac{\partial \Re f(z)}{\partial \theta} \quad \text{for} \quad |z| = 1
\]

we see, under the hypothesis,

\[
\Im \{zf'(z)\} = -\frac{\partial \Re (e^{i\theta})}{\partial \theta} > 0 \quad \text{for} \quad \theta_{2k-1} < \theta < \theta_{2k},
\]

\[
\Im \{zf'(z)\} = -\frac{\partial \Re (e^{i\theta})}{\partial \theta} < 0 \quad \text{for} \quad \theta_{2k} < \theta < \theta_{2k+1},
\]

\[
s = 1, 2, \ldots , \rho, \theta_j = \theta_1 + \pi, \theta_{2\rho+1} = \theta_1 + 2\pi.
\]

Hence \( f(z) \in F(p) \) if and only if \( zf'(z) \in D(p) \).

Theorem 2. Let

\[
f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n
\]

be a function of the set \( F(p) \). Suppose that in addition to the \((q-1)\)th order critical points at \( z=0 \), the function \( f(z) \) has exactly \( p-q \) critical points \( \alpha_1, \alpha_2, \ldots , \alpha_{p-q} \) such that \( 0 < |\alpha_j| < 1, j = 1, 2, \ldots , p-q \). Then

\[
|a_n| \leq qC_n/n, \quad n = q + 1, q + 2, \ldots ,
\]

(4.2)

\[
|f(re^{i\theta})| \leq q \int_0^r \frac{F(r)}{r} dr, \quad \text{for} \quad r < 1,
\]

(4.3)

\[
|f'(re^{i\theta})| \leq qF(r)/r, \quad \text{for} \quad r < 1,
\]

(4.4)

where \( C_n \) and \( F(r) \) are defined by

\[
F(z) = \frac{z^q}{(1 - z)^{2p}} \prod_{j=1}^{p} \left( 1 + \frac{z}{|\beta_j|} \right) (1 + z|\beta_j|)
\]

(4.5)

\[
= z^q + \sum_{n=q+1}^{\infty} C_n z^n.
\]

Proof. Since \( f(z) \in F(p) \),

\[
\frac{1}{q} zf'(z) = z^q + \sum_{n=q+1}^{\infty} n a_n z^n \in D(p)
\]

by Lemma 3.

By using the main theorem we have (4.2) and (4.4). By integrating \( f'(z) \) along a radius we have, for \( z=te^{i\theta} \),
$|f(re^{i\theta})| = \left| \int_0^1 f'(z) dz \right| \leq \int_0^1 |f'(re^{i\theta})| \, dr \leq q \int_0^1 \frac{F(r)}{r} \, dr$

for $r < 1$,

which completes the proof.

5. Subclasses of $D(p)$ and $F(p)$.

Corollary 1. Let $f(z)$ in the form (3.1) be regular for $|z| \leq 1$ and assigned with the same zeros as in Theorem 1. Suppose that $f(z)$ satisfies one of the following conditions:

(i) $\Re \left[ \frac{|zf'(z)|}{f(z)} \right] > 0$ for $|z| = 1$,

(ii) $f(1) = \text{real, } f(-1) = \text{real}$ and $\Re f(e^{i\theta})$ changes sign $2p$ times on $|z| = 1$,

(iii) $f(z) \in T(p)$.

Then (3.2) and (3.3) hold.

Proof. (i) Since there exists at least one diametral line of $f(z)$ by Lemma 1', and since $f(z)$ is starlike of order $p$ in every direction by the fact that $\Re \left[ \frac{|zf'(z)|}{f(z)} \right] > 0$ on $|z| = 1$ and $f(z)$ has $p$ zeros in $|z| < 1$, $f(z)$ is evidently starlike of order $p$ in the direction of the above diametral line.

(ii) In this case the diametral line of $f(z)$ is evidently the real axis and is starlike of order $p$ in this direction by our hypothesis, which proves the corollary by using the main theorem.

(iii) This is a direct consequence of the preceding (ii).

Corollary 2. Let $f(z)$ in the form (4.1) be regular for $|z| \leq 1$ and assigned with the same critical points as in Theorem 2. Suppose that $f(z)$ satisfies one of the following conditions:

(i) $1 + \Re \left[ \frac{|zf''(z)|}{f'(z)} \right] > 0$ for $|z| = 1$.

(ii) $f'(1) = \text{real, } f'(-1) = \text{real}$ and $f(z)$ is convex of order $p$ in the direction of the imaginary axis.

(iii) In (4.1) the coefficients are all real and $f(z)$ is convex of order $p$ in the direction of the imaginary axis.

Then (4.2), (4.3), and (4.4) hold.

Proof. (i) By our hypothesis $zf''(z)$ has $p$ zeros in $|z| < 1$ and $\Re \left[ s^p \{zf'(z)\}' \{zf'(z)\} \right] > 0$ on $|z| = 1$. Hence $zf'(z)$ is starlike of order $p$ in every direction. Consequently $zf'(z) \in D(p)$ by Corollary 1 adopting (i). Accordingly $f(z) \in F(p)$ by Lemma 3.

(ii) By our hypothesis $-\partial \Re f(z)/\partial \theta$ changes sign $2p$ times on $|z| = 1$. Accordingly $\Im \{zf'(z)\}$ changes sign $2p$ times on $|z| = 1$ by...
Lemma 3. And $1f'(1)=\text{real}$, $(-1)f'(-1)=\text{real}$. Hence $zf''(z)\in D(p)$. Consequently $f(z)\in F(p)$.

(iii) This is a special case of (ii).

REFERENCES


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