3. G. H. Hardy, *Notes on some points in the integral calculus*, Messenger of Mathematics vol. 58 (1929) pp. 50–52.

4. Einar Hille and J. D. Tamarkin, *On summability of Fourier series*. II, Ann. of Math. (2) vol. 34 (1933) pp. 329–348.

5. S. Verblunsky, *On some classes of Fourier series*, Proc. London Math. Soc. vol. 33 (1932) pp. 287–327.

6. F. H. Young, *A matrix transformation of Fourier coefficients*, Thesis, University of Oregon, 1950.

7. Antoni Zygmund, *Trigonometrical series*, Warsaw-Lwów, 1935, 331 pp.

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A REMARK ON LINEAR ELLIPTIC DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Consider a linear partial differential expression

$$L(u) = \sum_{i,k} a_{ik}(x) \frac{\partial u}{\partial x_i \partial x_k} + \sum_i b_i(x) \frac{\partial u}{\partial x_i}$$

with no term $c(x)u$. The coefficients a_{ik} and b_i are suppose to be continuous in an open connected set R of x-space, $x=(x_1, \cdots, x_n)$. Let x^0 denote a point on the boundary of R which has the property that R contains the interior of a hypersphere $|x-x^*|<r_0$ with x^0 on its boundary. Suppose that the coefficients are continuous at $x=x^0$ also. Let, finally, L be elliptic in $R+x^0$ such that the quadratic form

$$\sum a_{ik}(x)\lambda_i\lambda_k$$

is positive definite in each point of $R+x^0$.

This note contains a simple proof of the following:

THEOREM. *Suppose that $u=u(x)$ is of class C'' in R and that $u \geqq 0$, $L(u) \leqq 0$ in R. If the limit value of u at $x=x^0$ is zero, then either the normal derivative du/dn at $x=x^0$, understood as the limit inferior of $\Delta u/\Delta n$, is >0 or $u \equiv 0$ in R.*

Special cases of the theorem have been known for a long time. It contains, in particular, the fact that Green's function of L has a positive normal derivative along the boundary if the boundary is sufficiently smooth.

To prove the theorem we note first that $u \geqq 0$ in R and $u(x^0)=0$ trivially implies $du/dn \geqq 0$. The hypotheses that $u \geqq 0$ in R and $L(u) \leqq 0$ in R imply that either $u>0$ in R or $u \equiv 0$ in R. This follows from

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the sharp maximum-minimum-theorem.[1] It suffices to prove that $du/dn>0$ at x^0 if $u>0$ in R. Consider the sphere mentioned in the hypothesis. It may be assumed that its boundary has no other point in common with the boundary of R than x^0. Otherwise a second sphere which is internally tangent to the first one at $x=x^0$ would satisfy this condition. We choose its center as origin of the coordinate-system and we set $r=|x|$. Consider the closed spherical shell S: $r_0/2\leqq r\leqq r_0$ where r_0 denotes the radius of the sphere.[2] Obviously u is continuous in S, and

$$\begin{aligned} u &\geqq 0 \quad \text{on} \quad r = r_0, \\ u &= 0 \quad \text{at the point } x^0 \text{ of} \quad r = r_0, \\ u &> 0 \quad \text{on} \quad r = r_0/2. \end{aligned} \tag{1}$$

In my proof of the extremum-theorem I considered the auxiliary function

$$h(x) = e^{-ar^2} - e^{-ar_0^2}.$$

It has the property that $h>0$, $r<r_0$, and that

$$L(h) > 0, \qquad r_0/2 < r < r_0, \tag{2}$$

if the constant a is chosen sufficiently large. The reader can easily verify this fact himself if he uses the ellipticity of L and the continuity of the coefficients in the closed region S. h is of class C'' in S, and

$$h = 0 \quad \text{on} \quad r = r_0. \tag{3}$$

The function

$$v = u - \epsilon h, \qquad \epsilon > 0,$$

is of class C'' in the interior of S and continuous in S. Moreover, by (1) and (3),

$$v \geqq 0 \quad \text{on} \quad r = r_0. \tag{4}$$

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[1] E. Hopf, *Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus*, Sitzungsberichte der Berliner Akademie der Wissenschaften vol. 19 (1927) pp. 147–152.

[2] I owe the idea of using this type of region to my colleague D. Gilbarg who used it in a special case in order to prove the uniqueness of free boundary flow under more general conditions than considered hitherto. He considers a special differential equation $L(u)=0$ and uses a special solution h, $L(h)=0$, as an auxiliary function. See his paper *Uniqueness of axially symmetric flows with free boundaries*, Journal of Rational Mechanics and Analysis vol. 1 (1952) pp. 309–320, in particular pp. 314–315.

If the constant ϵ is chosen sufficiently small, then, by the third property (1),

$$v \geqq 0 \quad \text{also} \quad \text{on} \quad r = r_0/2. \tag{5}$$

By hypothesis, $L(u) \leqq 0$ in S, and by (2),

$$L(v) < 0, \qquad r_0/2 < r < r_0. \tag{6}$$

(4), (5), and (6) imply that $v \geqq 0$ holds in the whole of S. This follows again from the sharp extremum theorem or, this time more simply, from the more elementary fact that v cannot have a negative minimum in the interior of S. But $v \geqq 0$ in S and $v = 0$ at $x = x^0$ (see (3) and the second property (1)) imply that

$$\frac{dv}{dn} = \frac{du}{dn} - \epsilon \frac{dh}{dn} \geqq 0.$$

dh/dn is evidently >0. Hence $du/dn > 0$, q.e.d.

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