Consider a linear partial differential expression

\[ L(u) = \sum_{i,k} a_{ik}(x) \frac{\partial u}{\partial x_i \partial x_k} + \sum_i b_i(x) \frac{\partial u}{\partial x_i} \]

with no term \( c(x)u \). The coefficients \( a_{ik} \) and \( b_i \) are supposed to be continuous in an open connected set \( R \) of \( x \)-space, \( x = (x_1, \ldots, x_n) \). Let \( x^0 \) denote a point on the boundary of \( R \) which has the property that \( R \) contains the interior of a hypersphere \( |x - x^*| < r_0 \) with \( x^0 \) on its boundary. Suppose that the coefficients are continuous at \( x = x^0 \) also. Let, finally, \( L \) be elliptic in \( R + x^0 \) such that the quadratic form

\[ \sum a_{ik}(x) \lambda_i \lambda_k \]

is positive definite in each point of \( R + x^0 \).

This note contains a simple proof of the following:

**Theorem.** Suppose that \( u = u(x) \) is of class \( C^1 \) in \( R \) and that \( u \geq 0 \), \( L(u) \leq 0 \) in \( R \). If the limit value of \( u \) at \( x = x^0 \) is zero, then either the normal derivative \( du/dn \) at \( x = x^0 \), understood as the limit inferior of \( \Delta u/\Delta n \), is \( >0 \) or \( u \equiv 0 \) in \( R \).

Special cases of the theorem have been known for a long time. It contains, in particular, the fact that Green's function of \( L \) has a positive normal derivative along the boundary if the boundary is sufficiently smooth.

To prove the theorem we note first that \( u \geq 0 \) in \( R \) and \( u(x^0) = 0 \) trivially implies \( du/dn \geq 0 \). The hypotheses that \( u \geq 0 \) in \( R \) and \( L(u) \leq 0 \) in \( R \) imply that either \( u \) is positive or \( u \equiv 0 \) in \( R \). This follows from

Received by the editors February 18, 1952.
the sharp maximum-minimum-theorem.\footnote{E. Hopf, Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus, Sitzungsberichte der Berliner Akademie der Wissenschaften vol. 19 (1927) pp. 147–152.} It suffices to prove that $du/dn > 0$ at $x^0$ if $u > 0$ in $R$. Consider the sphere mentioned in the hypothesis. It may be assumed that its boundary has no other point in common with the boundary of $R$ than $x^0$. Otherwise a second sphere which is internally tangent to the first one at $x = x^0$ would satisfy this condition. We choose its center as origin of the coordinate-system and we set $r = |x|$. Consider the closed spherical shell $S$: $r_0/2 \leq r \leq r_0$ where $r_0$ denotes the radius of the sphere.\footnote{I owe the idea of using this type of region to my colleague D. Gilbarg who used it in a special case in order to prove the uniqueness of free boundary flow under more general conditions than considered hitherto. He considers a special differential equation $L(u) = 0$ and uses a special solution $h$, $L(h) = 0$, as an auxiliary function. See his paper Uniqueness of axially symmetric flows with free boundaries, Journal of Rational Mechanics and Analysis vol. 1 (1952) pp. 309–320, in particular pp. 314–315.} Obviously $u$ is continuous in $S$, and

\begin{align*}
  u &\geq 0 \quad \text{on} \quad r = r_0, \\
  u &= 0 \quad \text{at the point} \quad x^0 \quad \text{of} \quad r = r_0, \\
  u &> 0 \quad \text{on} \quad r = r_0/2.
\end{align*}

(1)

In my proof of the extremum-theorem I considered the auxiliary function

\[ h(x) = e^{-ar^2} - e^{-ar_0^2}. \]

It has the property that $h > 0$, $r < r_0$, and that

\[ L(h) > 0, \quad r_0/2 < r < r_0, \]

(2)

if the constant $a$ is chosen sufficiently large. The reader can easily verify this fact himself if he uses the ellipticity of $L$ and the continuity of the coefficients in the closed region $S$. $h$ is of class $C''$ in $S$, and

\[ h = 0 \quad \text{on} \quad r = r_0. \]

(3)

The function

\[ v = u - \epsilon h, \quad \epsilon > 0, \]

is of class $C''$ in the interior of $S$ and continuous in $S$. Moreover, by (1) and (3),

\[ v \geq 0 \quad \text{on} \quad r = r_0. \]

(4)
If the constant \( \varepsilon \) is chosen sufficiently small, then, by the third property (1),

\[
v \geq 0 \text{ also on } r = r_0/2.
\]

By hypothesis, \( L(u) \leq 0 \) in \( S \), and by (2),

\[
L(v) < 0, \quad r_0/2 < r < r_0.
\]

(4), (5), and (6) imply that \( v \geq 0 \) holds in the whole of \( S \). This follows again from the sharp extremum theorem or, this time more simply, from the more elementary fact that \( v \) cannot have a negative minimum in the interior of \( S \). But \( v \geq 0 \) in \( S \) and \( v = 0 \) at \( x = x^0 \) (see (3) and the second property (1)) imply that

\[
\frac{dv}{dn} = \frac{du}{dn} - \varepsilon \frac{dh}{dn} \geq 0.
\]

\( dh/dn \) is evidently \( > 0 \). Hence \( du/dn > 0 \), q.e.d.

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