OPERATOR ALGEBRAS AND DUAL SPACES

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1. An isomorphism theorem. In this paper we discuss a particular method of introducing a convolution product into certain topological linear spaces so that they become algebras. The space in each case will be essentially the dual space of a linear space of continuous functions, and arises in the study of general group algebras and in the theory of distributions.

Let G be a topological group, written additively but not necessarily abelian. Let $C^*(G)$ be the linear space of all complex-valued continuous functions on G. For $x \in G$ let U_x be the left translation operator which sends a function ϕ into the function $U_x\phi = \phi_x$ whose value at t is $\phi(x+t)$. Let X be a subspace of $C^*(G)$, topologized by a locally convex topology τ. Let $L(X)$ be the space of linear functionals on X which are τ continuous on all τ bounded sets of X. Let $B(X)$ be the algebra of linear operators on X which are τ bounded and τ continuous on the τ bounded sets of X. These contain the usual dual space X' and endomorphism algebra $E(X)$ and in certain important cases coincide with them, for example if τ is a norm topology. The operators U_x form a group G_0 of operators on $C^*(G)$ anti-isomorphic to G. If X is invariant under G_0 and τ suitable, we may regard G_0 as part of $B(X)$. In this case, let $B^0(X)$ be the set of all T in $B(X)$ which commute with G_0. $B^0(X)$ is evidently a subalgebra of $B(X)$. Our first theorem shows that under certain conditions $L(X)$ and $B^0(X)$ are isomorphic as linear spaces, thus permitting us to introduce an algebraic structure into $L(X)$.

We first define two fundamental mappings between operators and functionals. For any operator T in $B(X)$, let F be a functional on X defined by

$$F(\phi) = T(\phi)(0). \tag{1}$$

For a functional F in $L(X)$, let T be an operator on X defined by

$$T(\phi) = \psi \quad \text{where} \quad \psi(x) = F(U_x\phi). \tag{2}$$

THEOREM 1. *If the functional F defined by* (1) *is in $L(X)$ for each T in $B^0(X)$, and if T defined by* (2) *is in $B(X)$ for each F in $L(X)$, then* (1) (*or* (2)) *defines an isomorphism of $B^0(X)$ and $L(X)$ as linear spaces.*

Presented to the Society, April 27, 1951; received by the editors April 3, 1951 and, in revised form, January 11, 1952.

We need show only that (1) and (2) are inverse, and that T defined in (2) lies in the subalgebra $B^0(X)$. For the latter, $TU_y(\phi)(x) = F(U_xU_y\phi) = F(U_{y+x}\phi) = \psi(y+x) = U_y(\psi)(x) = U_yT(\phi)(x)$. Since this holds for all x and ϕ, $TU_y = U_yT$, and $T \in B^0(X)$. Given T in $B^0(X)$, define F by (1), and then T' by (2). Then, $T'(\phi)(x) = F(U_x\phi) = T(U_x\phi)(0) = U_xT(\phi)(0) = T(\phi)(x)$, and $T' = T$. Thus, (1) is a one-to-one embedding of $B^0(X)$ in $L(X)$. Given F in $L(X)$, define T by (2), and F' by (1). For any ϕ, $F'(\phi) = T(\phi)(0) = F(U_0\phi) = F(\phi)$, and $F' = F$.

Since $B^0(X)$ is also an algebra, the operation of operator multiplication induces a multiplication of functionals in the space $L(X)$. Given two functionals F_1 and F_2, let T_1 and T_2 be their images in $B^0(X)$. Then, $F_1 * F_2$ is defined to be the functional corresponding to T_1T_2. Under $*$, $L(X)$ becomes an algebra isomorphic to $B^0(X)$. The identity operator I corresponds to the functional δ defined by $\delta(\phi) = I(\phi)(0) = \phi(0)$; this "dirac function" is then the unit for $L(X)$. The operation $*$ can be given a more explicit description. For the value of F at ϕ, let us introduce the alternate notation $F(\phi(t)dt)$. Then, the function ψ of (2) may be given by $\psi(x) = F(\phi(z+t)dt)$ and $(F_1 * F_2)(\phi)$ is readily seen to be $F_1(F_2(\phi(x+t)dt)dx)$. The $*$ product can thus be considered as a generalization of the familiar convolution of functions and of measures.

The remaining sections will deal with special choices of X and τ under which the hypotheses of Theorem 1 may be established. Explicitly, it must be shown that F in (1) is τ continuous on τ bounded sets, that ψ in (2) lies in X, and that the mapping $\phi \to \psi$ is τ bounded and τ continuous on τ bounded sets. In addition, X must be invariant under left translation, and the operators U_x bounded and continuous.

2. **Spaces of continuous functions.** We distinguish certain special subspaces of $C^*(G)$. The first, $C^\#(G)$, is the set of functions ϕ having the property that for each, there is an open set $V \subset G$ such that ϕ is bounded on V and all of its translates. $C(G)$ is the set of all ϕ which are bounded on G, and $C_u(G)$ is the set of ϕ in $C(G)$ which are uniformly continuous on G. On each of these, several familiar topologies may be imposed. We shall use four, labeled as ω, κ, β, and σ. For convenience, set $\|\phi\|_S = \sup_{x \in S} |\phi(x)|$ for $S \subset G$. The topology ω is that of pointwise convergence on G, and is the smallest (weakest) of the topologies. The largest is σ which is that of uniform convergence on G. The former is applicable to C^* and its subspaces, and the latter to C. σ is a norm topology, defined by $\|\phi\|_G$ and, under it, C and C_u are Banach spaces. When G is locally compact, two additional topologies will be used. The κ topology is the usual one of

compact convergence, and is applicable to $C^*=C^\dagger$, and more especially to C. It may be defined by the collection of pseudonorms $\|\phi\|_K$ for compact sets $K\subset G$. It is not complete. The β topology, or "strict" topology, may be defined as follows. Let g be any non-negative function in $C(G)$ which vanishes at infinity, and set $\|\phi\|_g=\|\phi g\|_G$. β is the convex topology defined by this collection of pseudonorms. It is applicable to C and, as the following lemma shows, it has some of the better features of both σ and κ. It is similar to a topology introduced by Beurling [1].

LEMMA. *The β topology on $C(G)$ has the following properties*: (i) $\kappa\leqq\beta\leqq\sigma$, (ii) *$\beta$ and σ have the same bounded sets*, (iii) *β and κ coincide on σ bounded sets*, (iv) *β is a sequentially complete topology.*

Since $\|\phi\|_g\leqq\|\phi\|_G\|g\|_G$, β is smaller than σ. Given a compact set $K\subset G$, choose g_0 so that $g_0(x)=1$ on K. Then, $\|\phi\|_K\leqq\|\phi\|_{g_0}$ and $\kappa\leqq\beta$. Any uniformly bounded set is strictly bounded. Conversely, if $S\subset C(G)$ is strictly bounded but were not uniformly bounded, $\phi_n\in S$ and $x_n\in G$ could be chosen with $|\phi_n(x_n)|=\lambda_n$ and $\lim \lambda_n=\infty$. The sequence $\{x_n\}$ is convergent in G to infinity, and we may choose g so that $g(x_n)=\lambda_n^{-1/2}$. Then, $\|\phi_n\|_g\geqq\lambda_n^{1/2}$, contradicting the strict boundedness of S. To prove (iii), let S be a subset of $C(G)$ such that $\|\phi\|_G\leqq M$ for all $\phi\in S$, and let ϕ_0 be in the κ closure of S. Given $\epsilon>0$ and any g choose K so that $\|g\|_{G-K}<\epsilon$. Then,

$$\begin{aligned}\|\phi-\phi_0\|_g &\leqq \|\phi-\phi_0\|_K\|g\|_G+\|\phi-\phi_0\|_G\|g\|_{G-K}\\ &\leqq \|\phi-\phi_0\|_K\|g\|_G+\epsilon\|\phi-\phi_0\|_G.\end{aligned}$$

If $\phi\in S$ and $\|\phi-\phi_0\|_K<\epsilon$, then

$$\|\phi-\phi_0\|_g\leqq \epsilon\|g\|_G+(M+\|\phi_0\|_G)\epsilon$$

and ϕ_0 lies in the strict closure of S. Finally, a β Cauchy sequence is β bounded and hence σ bounded, and therefore κ and hence β convergent to a member of $C(G)$.

It is evident from the above that a sequence $\{\phi_n\}$ is β convergent if and only if κ convergent and uniformly bounded. Briefly, β may be described as the convex topology on $C(G)$ obtained from the pseudotopology of κ convergent uniformly bounded sets.

We shall set $\|F\|=\sup|F(\phi)|$, taken over all ϕ in X with $\|\phi\|_G\leqq 1$. Since a uniformly bounded set is bounded in each of the topologies, $\|F\|$ is finite for every F in $L(X)$. Similarly, if $T\in B(X)$, we set $\|T\|=\sup\|T(\phi)\|_G$, taken as above. We cannot immediately conclude that $\|T\|<\infty$, although this follows for $T\in B^0(X)$ from the next theorem.

THEOREM 2. *Under the convolution product* $*$ *and the norm* $\| \ \|$, *the spaces* $L(X)$ *become topological algebras in the following cases*: (i) $X = C^{\#}(G)$, $\tau=\omega$, (ii) $X=C(G)$, $\tau=\kappa$ *or* β, (iii) $X=C_u(G)$, $\tau=\sigma$.

We shall use Theorem 1 to establish a norm preserving isomorphism between $B^0(X)$ and $L(X)$ in each case. We first observe that each X is invariant under U_x. Moreover, $\|U_x\phi\|_S=\|\phi\|_{x+S}$ and $\|U_x\phi\|_g = \|\phi\|_h$ where $h=U_{-x}g$. Thus, in each case, the operators U_x lie in $B(X)$. Assume now that $T\in B^0(X)$ and define F by (1); $F(\phi) = T(\phi)(0)$. Since T is τ continuous on τ bounded sets and since $\omega\leqq\tau$, F is in $L(X)$. Consider the mapping $x\rightarrow U_x\phi$ of G into X. If ϕ is unbounded, but lies in $C^{\#}$, there is a neighborhood V of 0 in G such that the set of all $U_x\phi$ for $x\in V$ is an ω bounded subset. Since ϕ itself is continuous, the mapping $x\rightarrow U_x\phi$ is continuous at 0. If $F\in L(X)$ in case (i), F is continuous on ω bounded sets, and $\psi(x) = F(U_x\phi)$ is continuous for $x\in V$. Moreover, F is bounded on ω bounded sets so that $\psi(x)$ is bounded on V. Similarly, ψ is bounded and continuous on each translate of V, and thus lies in $C^{\#}$. We turn next to case (ii). Since $\phi\in X$ is now bounded, the set of all $U_x\phi$ for $x\in G$ is a σ bounded set, and thus κ and β bounded. Since κ and β coincide on σ bounded sets, we can discuss these together. If $K\subset G$ is compact, then ϕ is uniformly continuous on K and the mapping $x\rightarrow U_x\phi$ is continuous; as before, for any F in $L(C)$, $\psi(x)$ is continuous, and since $|\psi(x)|\leqq\|F\|\|\phi\|_G$, ψ is bounded and therefore lies in $C(G)=X$. Finally, in case (iii) ϕ is uniformly continuous on all of G and $x\rightarrow U_x\phi$ is uniformly σ continuous so that $\psi(x)$ is uniformly continuous and therefore in C_u. In each case, we have verified that (2) defines a linear transformation T of X into itself. It remains to show that T is appropriately continuous and bounded. Since $\|T(\phi)\|_G=\|\psi\|_G\leqq\|F\|\ \|\phi\|_G$ we see that, in all cases, $\|T\|\leqq\|F\|<\infty$. Since T is then σ continuous on σ bounded sets, the proof of case (iii) is complete. Case (i) is also easily disposed of. For fixed x, FU_x is continuous on ω bounded sets, and it then follows at once that T is ω continuous on ω bounded sets. Similarly, since F must be bounded on ω bounded sets, T carries ω bounded sets into ω bounded sets. Case (ii) requires slightly more argument. Here $X=C(G)$ and $\tau=\kappa$ or β. We discuss κ first. Let S be a κ bounded set in X containing 0, and let a compact set $K\subset G$ and $\epsilon>0$ be given. Let $S^*=\{\text{all } U_x\phi \mid x\in K, \phi\in S\}$. S^* is also κ bounded. Since $F\in L(X)$ is κ continuous on S^*, choose K_1 and δ so that $\phi\in S^*$ and $\|\phi\|_{K_1}<\delta$ imply $|F(\phi)|<\epsilon$. Set $K_2=K+K_1$. Then, if $x\in K$, $\|U_x\phi\|_{K_1}\leqq\|\phi\|_{K_2}$ so that $\phi\in S$ and $\|\phi\|_{K_2}<\delta$ imply that $|F(U_x\phi)|<\epsilon$ for all $x\in K$. Rephrased, this states that if $\phi\in S$ and $\|\phi\|_{K_2}<\delta$, then $\|\psi\|_K<\epsilon$, proving that T is

κ continuous on κ bounded sets. We have also proved that the image of S is bounded, since $|F(\phi)| \leqq A$ holds for all $\phi \in S^*$, so that if $\phi \in S$, $\|\psi\|_K \leqq A$. The argument for $\tau = \beta$ is implicit in the discussion above, since the set S being β bounded is σ bounded, and so is S^*, and β and κ agree on such sets. Finally, to show that the correspondence established in (1) and (2) is norm preserving, we note that we have shown above that $\|T\| \leqq \|F\|$. Since $|F(\phi)| = |T(\phi)(0)| \leqq \|T(\phi)\|_G$, we also have $\|F\| \leqq \|T\|$.

It perhaps should be be pointed out that the space $L(X)$ in case (ii) contains an image of the usual L^1 algebra of G. If $f \in L^1(G)$, then the functional F defined by $F(\phi) = \int_G f\phi$ is in $L(C(G))$ if τ is chosen as β, although not if $\tau = \kappa$. This approach, using case (iii), has been discussed elsewhere [2].

3. **The space of distributions.** Let G be the additive group of reals, with the usual locally compact topology. Let $\mathcal{E}$ be the subspace of $C^*(G)$ composed of the infinitely differentiable functions, and let $\mathcal{D}$ be the subspace of those ϕ which have compact support, that is, vanish off a compact set. On $\mathcal{E}$ we impose the topology τ generated by the pseudonorms $\|\phi\|_{K,p} = \|D^p\phi\|_K$ where D is the differentiation operator, and K is a compact set. Clearly τ-lim $\phi_\alpha = 0$ means merely that κ-lim $D^p\phi_\alpha = 0$ for $p = 0, 1, \cdots$. Let $\mathcal{E}'$ and $E(\mathcal{E})$ be the usual dual space and endomorphism algebra of $\mathcal{E}$; $E^0(\mathcal{E})$ is again the subalgebra of those T in $E(\mathcal{E})$ which commute with the translation operators U_x.

THEOREM 3. *$\mathcal{E}'$ and $E^0(\mathcal{E})$ are isomorphic under the mappings* (1) *and* (2).

We first observe that if T is in $E^0(C^*)$ where C^* has the κ topology, and if T leaves $\mathcal{E}$ invariant and commutes with D, T is in $E^0(\mathcal{E})$. As before, since $\omega \leqq \tau$, the functional F defined by (1) is continuous and hence in $\mathcal{E}'$. Given F in $\mathcal{E}'$ and defining T by (2), T is easily seen to belong to $E^0(C^*)$. With $\psi(x) = F(U_x\phi)$, we have $D\psi(x) = \lim_{h\to 0} F(h^{-1}(U_{x+h} - U_x)\phi)$. For each p, $D^p\phi$ is continuous, and hence uniformly continuous on compact sets, and thus τ-$\lim_{h\to 0} h^{-1}(U_{x+h} - U_x)\phi = U_xD\phi$. Since F is τ continuous, $D\psi(x) = F(U_xD\phi)$, and by induction, $D^p\psi(x) = F(U_xD^p\phi)$. This proves both that ψ lies in $\mathcal{E}$ so that T leaves $\mathcal{E}$ invariant, and that T commutes with D. By the remark above, T is in $E^0(\mathcal{E})$.

The algebra $E^0(\mathcal{E})$ may be considered as the set of all κ continuous operators on C^* which carry $\mathcal{E}$ into itself and commute with D and with translations. It contains the subalgebra of all differential operators with constant coefficients and of finite order, but does not contain such operators as x^2D. It also contains such integral transforms as $T(\phi)=\psi$ where

$$\psi(x) = \int_{-\infty}^{\infty} H(x-t)\phi(t)dt$$

where $H\in\mathcal{D}$. More generally, the theorem asserts that the general operator T has the representation $T(\phi)=\psi$ with $\psi(x)=F(U_x\phi)$ for a unique F in $\mathcal{E}'$. This latter space is exactly the space of distributions having compact support (see [3, chap. 3]). As before, the multiplication of operators in $E^0(\mathcal{E})$ induces a convolution product into $\mathcal{E}'$ so that it too becomes an algebra. It is readily seen that this is the same product as that introduced in a different way by Schwartz. That portion of the theory of distributions relating to $\mathcal{E}'$ can be transferred with some advantage to the operator algebra $E^0(\mathcal{E})$. In place of the "derivative" of a distribution, the operator D itself is present; if F is the distribution corresponding to the operator T, the derivative F' corresponds to the operator $-DT$.

Turning now to the space $\mathcal{D}$, the importance of this subspace of $\mathcal{E}$ lies partly in the fact that, on it, D is a one-to-one operator, whose range is the set of all ϕ with $\int_{-\infty}^{\infty}\phi=0$. We can consider $\mathcal{D}$ under the relative τ topology or under the Schwartz topology τ' in which a convergent sequence must have a common supporting set. The dual space $\mathcal{D}'$ is the space of all distributions; let $E(\mathcal{D})$ be the corresponding endomorphism algebra. The natural mappings (1) and (2) do not set up an isomorphism of $\mathcal{D}'$ and $E^0(\mathcal{D})$; instead, $E^0(\mathcal{D})$ is embedded isomorphically in a proper part of $\mathcal{D}'$. For example, the distribution 1 corresponds to the functional F where $F(\phi)=\int_{-\infty}^{\infty}\phi$. The corresponding operator defined by (2) sends ϕ into ψ, with $\psi(x)=F(\phi)$ for all x; ψ is a constant function, and lies in $\mathcal{E}$ but not $\mathcal{D}$. It is possible to obtain a space of operators which is isomorphic to the whole space $\mathcal{D}'$. Consider $E^0(\mathcal{D};\,\mathcal{E})$, the set of all continuous linear transformations of $\mathcal{D}$ into $\mathcal{E}$ which commute with U_x for $x\in G$. The previous type of argument shows that (1) and (2) set up an isomorphism between $\mathcal{D}'$ and $E^0(\mathcal{D};\,\mathcal{E})$. The convolution of distributions can now be discussed in terms of the ordinary product of operators. Given T_1 and T_2 in $E^0(\mathcal{D};\,\mathcal{E})$, the product T_1T_2 is an operator whose domain is $\mathcal{D}\cap T_2^{-1}(\mathcal{D})$. It corresponds to a distribution only if this domain is $\mathcal{D}$ (and thus if $T_2(\mathcal{D})\subset\mathcal{D}$) or, more generally, if the domain of T_1 can be expanded to include the range of T_2. For example, we recall that a distribution with compact support corresponds to an operator T mapping $\mathcal{E}$ into $\mathcal{E}$, and $\mathcal{D}$ into $\mathcal{D}$. It is therefore possible to form the convolution of an arbitrary distribution and one with compact support, in either order. (See [**3**, chap. 6].) In going from an operator to the corresponding distribution, only the action of the operator on $\mathcal{D}$ is considered; in this contraction, some properties of the operator algebra are lost, in particular convolution of distributions fails to be associative, although the product of operators of course is [**3**, chap. 6, §5]. This suggests that it would perhaps be fruitful to study the general algebra of operators T whose domains are subspaces of $\mathcal{E}$ containing $\mathcal{D}$, and whose ranges lie in $\mathcal{E}$.

References

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