ON THE FACTORIZATION OF SQUAREFREE INTEGERS

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In recent years several papers [1; 3; 4; 5; 6; 7; 9; 10; 11] have appeared dealing with the problem of "Factorisatio numerorum," the number $f(n)$ of representations of an integer n as an ordered product of factors greater than 1. As a result, the basic combinatorial properties of $f(n)$ and the asymptotic behavior of its summatory function are well known. In this paper, I determine the asymptotic behavior of $f(n)$ itself for squarefree n and use the result to determine a "normal" order of $f(n)$ for all n.

If n is written as a product of powers of distinct primes p_i, then $f(n)$ is evidently a symmetric function of the exponents of the p_i's. If n is squarefree, all the exponents are 1 and $f(n)$ can be considered a function of the single variable r, the number of distinct prime factors of n. It is therefore convenient to define, for positive integral r:

$$h(r) = f(S_r),$$

where S_r represents any squarefree integer with r prime factors.

A (convergent) asymptotic expansion of $h(r)$ is given by:

THEOREM 1. (a) *For every positive integral r,*

$$h(r) = 2^{-1}r!(\log 2)^{-r-1}\{1 + R_r\}, \tag{1}$$

and the remainder R_r can be expressed in the form:

$$2\sum_{n=1}^{\infty} (\cos \theta_n)^{r+1} \cos \{(r+1)\theta_n\}, \tag{2}$$

where θ_n is defined by:

$$\cos \theta_n = \frac{\log 2}{2\pi n}\left\{1 + \left(\frac{\log 2}{2\pi n}\right)^2\right\}^{-1/2}. \tag{3}$$

(b) *For every positive integral r,*

$$| R_r | \leqq 2\zeta(r+1)\left(\frac{\log 2}{2\pi}\right)^{r+1} \tag{4}$$

where $\zeta(r+1)$ is the Riemann zeta function.

PROOF. Sen [10] proves the identity:[1]

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[1] The identity (5) is a special case of the general relation $f(n) = 2^{-1}\sum d_k(n)2^{-k}$, which relation does not seem to appear explicitly anywhere in the literature. $d_k(n)$ is the number of ordered factorizations of n into k factors greater than or equal to 1.

$$h(r) = \frac{1}{2}\sum_{n=0}^{\infty} n^r 2^{-n}. \tag{5}$$

Part (a) of Theorem 1 is merely the result of applying the Euler-MacLaurin formula to (5).

For this purpose, let $g_r(x) = 2^{-1}x^r 2^{-x}$ for x real. Then $g_r(x)$ and all its derivatives vanish at $+\infty$, are integrable from 0 to $+\infty$, and $g_r(x)$ and its derivatives up to order $r-1$ vanish at $x=0$. Since $g_r(n) = 2^{-1}n^r 2^{-n}$, the Euler-MacLaurin formula gives for every positive integral k:

$$\begin{aligned} h(r) = \int_0^\infty g_r(x)dx - \sum_{j=1}^{k} \frac{B_{2j}}{(2j)!} g_r^{(2j-1)}(0) \\ + \int_0^\infty p_{2k+1}(x) g_r^{(2k+1)}(x)dx, \end{aligned} \tag{6}$$

the notation being that of Knopp [8, pp. 524–526].

The first term on the right gives the dominant term in (1), since

$$\int_0^\infty g_r(x)dx = \frac{1}{2}\int_0^\infty x^r\, 2^{-x}dx = \frac{1}{2}\, r!\,(\log 2)^{-r-1}.$$

Set

$$I_{r,k} = \int_0^\infty p_{2k+1}(x) g_r^{(2k+1)}(x)dx.$$

Then

$$|I_{r,k}| \leqq 4(2\pi)^{-2k-1}\int_0^\infty |g_r^{(2k+1)}(x)|\, dx$$

since $|P_{2k-1}(x)| \leqq 4(2\pi)^{-2k-1}$ [8, p. 527].

From Leibnitz's rule, for any positive integral m:

$$g_r^{(m)}(x) = \frac{1}{2}\sum_{l=0}^{\min(m,r)} (-1)^{m-l} C_{m,l} C_{r,l}\, l!(\log 2)^{m-l} x^{r-l} 2^{-x}. \tag{7}$$

It immediately follows that:

$$|g_r^{(2k+1)}| \leqq \frac{1}{2}\sum_{l=0}^{\min(m,r)} C_{2k+1,l} C_{r,l}\, l!(\log 2)^{2k+1-l} x^{r-l} 2^{-x}.$$

If $f_j(n)$ be defined as the number of factorizations of n into j factors greater than n, then the general relation is a consequence of the identity (Strehler [11]) $d_k(n) = \sum_{j=1}^{k} C_{k,j} f_j(n)$.

When this is substituted into the inequality for $|I_{r,k}|$, and the integral evaluated, the result is:

$$
\begin{aligned}
|I_{r,k}| &\leqq \frac{1}{\pi}\frac{r!}{(\log 2)^r}\left(\frac{\log 2}{2\pi}\right)^{2k}\sum_{l=0}^{2k+1} C_{2k+1,l} \\
&= \frac{2}{\pi}\frac{r!}{(\log 2)^r}\left(\frac{\log 2}{\pi}\right)^{2k}
\end{aligned}
$$

where the expression on the right tends to zero, for fixed r, as k increases.

Since the remainder in (6) tends to zero as k increases, (6) may be written:

$$h(r) = \frac{1}{2} r!(\log 2)^{-r-1} - \sum_{j=1}^{\infty}\frac{B_{2j}}{(2j)!} g_r^{(2j-1)}(0).$$

From (7),

$$g_r^{(2j-1)}(0) = \frac{1}{2}(-1)^{2j-r-1}C_{2j-1,r}r!(\log 2)^{2j-r-1}.$$

Hence,

$$R_r = (-1)^r\sum_{j=1}^{\infty}\frac{B_{2j}}{(2j)!}C_{2j-1,r}(\log 2)^{2j}.$$

Now [8, p. 237],

$$\frac{B_{2j}}{(2j)!} = (-1)^{j-1}\frac{2}{(2\pi)^{2j}}\sum_{n=1}^{\infty}\frac{1}{n^{2j}}\cdot$$

Therefore,

$$
\begin{aligned}
R_r &= (-1)^{r+1}\sum_{j=1}^{\infty}\sum_{n=1}^{\infty} 2(-1)^j C_{2j-1,r}\left(\frac{\log 2}{2n\pi}\right)^{2j} \\
&= (-1)^{r+1}\sum_{j=1}^{\infty}\sum_{n=1}^{\infty} 2\cos\frac{\pi j}{2} C_{j-1,r}a_n^j
\end{aligned}
$$

where $a_n = (\log 2)/2n\pi$. Substitution of $i^j + (-i)^j$ for $2\cos(2^{-1}\pi j)$, interchanging of the order of summation, and use of the identity

$$\sum_{j=1}^{\infty} C_{j-1,r}y^j = y^{r+1}(1-y)^{-r-1}$$

leads to the expression:

$$R_r = (-1)^{r+1}\sum_{n=1}^{\infty} a_n^{r+1}\{(-i - a_n)^{-r-1} + (i - a_n)^{-r-1}\}$$

which, when put back into real form, is (2).

Part (b) of Theorem 1 follows from (2) and (3), for:

$$\begin{aligned} | R_r | &\leqq 2\sum_{n=1}^{\infty} | \cos \theta_n |^{r+1} \\ &\leqq 2\sum_{n=1}^{\infty}\left(\frac{\log 2}{2\pi n}\right)^{r+1} = 2\left(\frac{\log 2}{2\pi}\right)^{r+1}\zeta(r+1). \end{aligned}$$

Deviating slightly from Hardy and Ramanujan [2], I define a "normal" order of a numerical function $F(n)$ as a function $G(n)$ such that $F(n) \sim G(n)$ as n increases over a set that includes almost all integers ("almost all" in the sense of Hardy and Ramanujan [2]). Under this definition, a consequence of Theorem 1 is:

THEOREM 2. *A normal order of* $\log f(n)$ *is* $\log \log n \log \log \log n$.

PROOF. Let $r_1 = r_1(n)$ be the number of distinct prime factors of n, and $r_2 = r_2(n)$ be the total number of prime factors of n, distinct or not. Evidently, $\log h(r_1) \leqq \log f(n) \leqq \log h(r_2)$. By Theorem 1, $\log h(r) = r \log r\{1+o(1)\}$ as r increases. Hence,

$$r_1 \log r_1\{1 + o(1)\} \leqq \log f(n) \leqq r_2 \log r_2\{1 + o(1)\}.$$

Now r_1 and r_2 both have the normal orders $\log \log n$ [2, Theorems B′ and C′]. Therefore, for almost all n, $\log \log n \log \log \log n\{1+o(1)\} \leqq \log f(n) \leqq \log \log n \log \log \log n\{1+o(1)\}$ which is equivalent to Theorem 2.

A maximum order of a non-negative numerical function $F(n)$ defined over an infinite set of positive integers may be defined as a function $G(n)$ such that $\limsup_{n\to\infty} F(n)/G(n) = 1$.

THEOREM 3. *A maximum order of* $\log f(n)$ *for squarefree* n *is* $\log n$.

PROOF. Take n_k to be the product of the first k primes. Then $\log f(n_k) = \log h(k)$ and $\log f(n_k) \sim k \log k$, since $\log h(k) \sim k \log k$ by Theorem 1. By the prime-number theorem $k \log k \sim \log n_k$, hence

$$\log f(n_k) \sim \log n_k.$$

For any squarefree integer n, there is an n_k for which $f(n_k) = f(n)$, $n_k \leqq n$. Hence $\log f(n) = \log f(n_k) = \log n_k\{1+o(1)\} \leqq \log n\{1+o(1)\}$.

Theorem 3 may be compared with the fact (Hille [3], Ikehara

[5]) that the maximum order of $\log f(n)$ for all n is $\rho \log n$, where ρ is the unique positive number with the property $\zeta(\rho)=2$.

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