INACCESSIBLE BOUNDARY POINTS

A. W. GOODMAN

1. Introduction. The Riemann mapping theorem gives immediately the existence of a function $F(z)$, analytic in $E$, the circle $|z| < 1$, and mapping that region onto a region having inaccessible boundary points [3, pp. 179–200].

However, so far as the author is aware, there are no known formal expressions for an $F(z)$ of this type, and it is the purpose of the present work to remedy this defect by giving a class of examples of such functions. As a particular case we shall see that the function

\[(1.1) \quad F(z) = \frac{z}{\prod_{n=1}^{\infty} (1 - z^2 \cos (\pi/2(n+1/2)) + z^2)^{1/2n}}\]

maps $E$ onto a region $B_\alpha$ of the $w$-plane, for which all of the points $w > \delta$ are inaccessible boundary points.

2. The slit regions $B_\alpha$. We shall consider regions formed by deleting from the entire complex plane an infinite number of semi-infinite radial slits, symmetrically placed with respect to the real axis. Because of this symmetry it is sufficient to consider only the slits in the closed upper half-plane. Let $S_n$ denote the slit with end point at $w = \rho_n e^{i\phi_n}$ and suppose the subscripts so chosen that

\[(2.1) \quad \pi = \phi_0 > \phi_1 > \phi_2 > \cdots > \phi_n > \cdots > \phi_\infty = 0.\]

DEFINITION. A region $B_\alpha$ is said to be of type $S$ if it is formed as described above, and if in addition:

(a) There is a constant $m > 0$ such that

\[m \leq \rho_n < \infty, \quad n = 0, 1, 2, \cdots,\]

(b) \[\lim_{n \to \infty} \phi_n = 0,\]

(c) \[\rho_\infty = \lim \inf_{n \to \infty} \rho_n.\]

It is obvious that if $\rho_\infty < \infty$, then all of the points $w > \rho_\infty$ are inaccessible boundary points of $B_\alpha$. Condition (a) assures the existence of an $F(z)$, such that $F(0) = 0$, $F'(0) > 0$, and $F(z)$ maps $E$ conformally onto $B_\alpha$. Theorem 1 gives somewhat more information about $F(z)$.

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1 Numbers in brackets refer to the references at the end of the paper.
Theorem 1. For each region $B_*$ of type $S$ there are constants $c > 0, \theta_n,$

\begin{equation}
\pi > \theta_1 > \theta_2 > \cdots > \theta_n > \cdots > 0, \quad \theta_n \to \theta_* = 0,
\end{equation}

such that if

\begin{equation}
F(z) = \frac{cz}{\prod_{n=1}^{\infty} (1 - z^2 \cos \theta_n + z^2)^{\gamma_n}}
\end{equation}

where

\begin{equation}
\gamma_n \pi = \phi_{n-1} - \phi_n, \quad n = 1, 2, \cdots,
\end{equation}

then $F(z)$ maps $E$ conformally onto $B_*$ with $F(0) = 0$ and $F'(0) > 0.$

Conversely each function defined by (2.3) and (2.2) with

\begin{equation}
\sum_{n=1}^{\infty} \gamma_n = 1, \quad \gamma_n > 0; n = 1, 2, \cdots,
\end{equation}

maps $E$ conformally onto a region of type $S$ where the directions of the slits are determined by

\begin{equation}
\phi_n = \pi \sum_{j=n+1}^{\infty} \gamma_j.
\end{equation}

Proof. First note that each term of the product

\begin{equation}
P_n(z) \equiv (1 - z^2 \cos \theta_n + z^2)^{\gamma_n} = (1 - ze^{i\theta_n})^{\gamma_n}(1 - ze^{-i\theta_n})^{\gamma_n}
\end{equation}

is to be understood as that branch of the function for which $P_n(0) = 1.$ Let $E_n(\delta)$ denote the region obtained by deleting from $E$ the portion common to the circle $|z - e^{i\theta_n}| \leq \delta,$ and the portion common to the circle $|z - e^{-i\theta_n}| \leq \delta,$ $\delta > 0;$ and let $E(\delta)$ denote the intersection of the sets $E_n(\delta), n = 1, 2, \cdots.$ Then it is clear from (2.7) that in $E_n(\delta)$

\begin{equation}
\delta^{2\gamma_n} \leq |P_n(z)| \leq 4^{\gamma_n},
\end{equation}

\begin{equation}
-\gamma_n \pi < \arg P_n(z) < \gamma_n \pi,
\end{equation}

and hence for $0 < \delta < 1,$

\begin{equation}
|\log P_n(z)| \leq \gamma_n \{\log 4 - 2 \log \delta + \pi\}.
\end{equation}

From (2.5) it follows that in $E(\delta)$ the series

\begin{equation}
\sum_{n=1}^{\infty} \log P_n(z)
\end{equation}

converges uniformly and hence by the Weierstrass Theorem repre-
presents an analytic function in $E(\delta)$. Since $\delta$ can be taken arbitrarily small, the series (2.11) converges in $E$ and hence the product in (2.3) is also convergent in the same region. In fact, the product is uniformly convergent on any closed arc of $|z|=1$ which is free of the points $e^{\pm i\theta_n}$, $n = 1, 2, \ldots$.

Suppose now that we have a given fixed region $B_*$ of type $S$. By the Riemann mapping theorem, there is a unique function $F_*(z)$ mapping $E$ conformally on $B_*$ with $F_*(0) = 0$ and $F'_*(0) > 0$. From the symmetry of $B_*$, $F_*(z)$ is real on the real axis and further $r F'_*(r) > 0$ for $0 < r < 1$. Denote by $E_c$ the region obtained by cutting the open unit circle along the negative real axis, $-1 < r \leq 0$. Then $\arg F_*(z)$ is harmonic in $E_c$, and further, with a suitable determination, $-\pi \leq \arg F_*(z) \leq \pi$. A consideration of $1/F_*(z)$ shows that each of the points on the inverted slits $S_n^{-1} (w=\rho e^{-i\theta_n}$, $0 \leq \rho \leq \rho_n^{-1}$, $0 < \phi \leq \pi$) is an accessible boundary point and, therefore, the zero of $1/F_*(z)$ corresponding to the vertex of the sector defined by $S_n^{-1}$ and $S_{n-1}^{-1}$ is the image of a well-determined point $z = e^{i\theta_n}$ [3, pp. 189–192]. Even more, the Schwarz reflection principle shows that $z = e^{i\theta_n}$ is a simple zero of $(F_*(z))^{-1/\gamma_n}$. Thus the function $F_*(z)$ determines a set of arcs on the boundary of the upper half of the unit circle with end points $e^{i\theta_n}$ satisfying (2.2), such that for $\theta_{n+1} < \theta < \theta_n$

$$\arg F_*(e^{i\theta}) = \sum_{j=n+1}^{\infty} \gamma_j \pi = \phi_n, \quad n = 1, 2, \ldots.$$ 

For $\theta_1 < \theta < \pi$, $\arg F_*(z) = \pi$. Finally, if $\theta_\infty > 0$, then $\arg F_*(z)$ can be extended by continuity so that $\arg F_*(e^{i\theta}) = 0$ for $-\theta_\infty < \theta < \theta_\infty$. To see this last assertion, observe that any simple curve $\Gamma_w$ in $B_*$ joining $\rho_n e^{i\theta_n}$ and $\rho_n e^{-i\theta_n}$ is the image of some simple curve $\Gamma_z$ in $E$ joining $e^{i\theta_n}$ and $e^{-i\theta_n}$. Given $\epsilon > 0$, an appropriately chosen $\Gamma_w$ with $n$ sufficiently large determines a curve $\Gamma_n$, which, together with the arc $z = e^{i\theta}$, $-\theta_n \leq \theta \leq \theta_n$, determines a region in which $|\arg F_*(z)| \leq \phi_n < \epsilon$.

Thus $\arg F_*(z)$ is a harmonic function in $E_c$, continuous and bounded on the interior, constant on the boundary except for finite jumps of $\pm \gamma_n \pi$ at $\pm \theta_n$, and a jump of $2\pi$ at $z = 0$.

Next, with the values of $\theta_n$ just determined for $F_*(z)$, and with the associated values of $\gamma_n$, form the function $F(z)$ as in (2.3). If $u = \arg F(z)$ for $z$ in $E_c$, then $u$ is also harmonic there and

$$u = \arg z - \sum_{j=1}^{\infty} \gamma_j \arg (1 - z^2 \cos \theta_j + z^2)$$

(2.12)

$$= \sum_{j=1}^{\infty} \gamma_j \{\theta - \arg (1 - z^2 \cos \theta_j + z^2)\}.$$
For $z = e^{i\theta}$,

$$\arg (1 - z^2 \cos \theta + z^2) = \begin{cases} 
\theta, & \text{if } 0 \leq \theta < \theta_i, \\
\theta - \pi, & \text{if } \theta_i < \theta \leq \pi.
\end{cases}$$

Thus for the upper half of the boundary of $E_n$, $u$ is a step function, such that for $\theta_{n+1} < \theta < \theta_n$

$$u = \arg F(e^{i\theta}) = \sum_{j=n+1}^{\infty} \gamma_j \left\{ \theta - (\theta - \pi) \right\} = \phi_n.$$

Similar results hold for the lower half of the boundary of $E_n$. Finally when $z=r$, $-1 < r < 0$, $u = \pm \pi$ according as $z$ approaches the boundary from above or below.

Therefore $U = u_n - u$ is a bounded harmonic function, continuous in $E$ and zero on the boundary except for an infinite number of points which have at most two limit points. The conformality of $F(z)$ and $F_i(z)$ at the origin permit us to remove the slit, and make the same assertions about $U$ in $E$. It is then easy to see from the Poisson integral formula that $U \equiv 0$ [4, p. 321]. Thus $F(z)$ and $F_i(z)$ differ by at most a multiplicative positive constant. Thus if $c$ is chosen properly in (2.3), $F(z)$ maps $E$ onto the region $B_i$. It is now easy to see that $\theta_\infty = 0$, for otherwise the arc

$$z = e^{i\theta}, \quad -\theta_\infty < \theta < \theta_\infty,$$

would go into a doubly covered slit on the real axis consisting of accessible boundary points. This is indeed a possibility, but we have excluded this possibility by condition (c) of the definition of the region $E_n$. If, in (2.3), $\theta_\infty > 0$ then $F(z)$ will map $E$ onto a region with accessible boundary points on the positive real axis whether or not it has inaccessible boundary points. We have defined $B_i$ in such a way as to exclude this occurrence in order to simplify the presentation.

The second part of Theorem 1 will be a trivial consequence of the preceding material as soon as we show that the accessible boundary points of $B_i$ form a one-to-one image of the arc $|z| = 1$, $z \neq 1$, under $F(z)$, i.e. as soon as we show that each slit $S_j$ is doubly covered. Recalling (2.5) it is easy to see that

$$z F'(z) = \sum_{j=1}^{\infty} \frac{\gamma_j (1 - z^2)}{1 - z^2 \cos \theta_j + z^2}.$$

Thus $F'(-1) = 0$, and if $z = 1$ is a point of regularity of $F(z)$, i.e. if $\theta_\infty \neq 0$, then $F'(1) = 0$. For $z = e^{i\theta}$, $\theta_{n+1} < \theta < \theta_n$.
\[ \frac{zF'(z)}{F(z)} = -i \sin \theta \left\{ \sum_{j=1}^{n} \frac{\gamma_j}{\cos \theta - \cos \theta_j} - \sum_{j=n+1}^{\infty} \frac{\gamma_j}{\cos \theta_j - \cos \theta} \right\} \]

(2.16)

\[ = -i \sin \theta \left\{ I(\theta) - D(\theta) \right\} \]

where \( I(\theta) \) is an increasing function, tending to \( \infty \) as \( \theta \to \theta_n \), and \( D(\theta) \) is a decreasing function, decreasing from \( \infty \) at \( \theta = \theta_{n+1} \). Thus in each arc \( \theta_{n+1} < \theta < \theta_n \), \( n = 1, 2, \cdots, F'(z) \) has a simple zero which we denote by \( z = e^{i\theta_n} \), and \( F(e^{i\theta_n}) = \rho_n e^{i\theta_n} \), a slit end point.

In case \( \theta_n > 0 \), \( z = 1 \) furnishes the simple zero of \( F'(z) \) for the arc \( -\pi < \theta < \pi \), while \( z = -1 \) is always the simple zero for \( \theta_1 < \theta < 2\pi - \theta_1 \). The slits \( S_j \) are doubly covered, and thus Theorem 1 is proved.

We observe that if the region \( B_s \) is not assumed to be symmetrical, then (2.3) is replaced by

\[ F(z) = \frac{cz}{\prod_{n=1}^{\infty} \left( 1 - z e^{i\theta_n} \right)^{\gamma_n}} \]

(2.17)

where now

\[ \sum_{n=1}^{\infty} \gamma_n = 2, \quad \gamma_n > 0. \]

(2.18)

If \( B_s \) has only \( m \) slits, then in (2.17) and (2.18) the product and sum have exactly \( m \) terms. In the still more special case that the \( e^{i\theta_n} \) are taken as the \( m \)th roots of unity, and the \( \gamma_n \) all equal, we obtain the well known

\[ f(z) = \frac{cz}{\left( 1 - z^m \right)^{2/m}}, \]

which maps the unit circle on the \( w \)-plane with radial slits whose end points are the vertices of a regular \( m \)-gon.

Perhaps a more intuitive approach to (2.17) and (2.3) would have been through Alexander's theorem [1; 5, pp. 11–15], which states that if

\[ F(z) = zf'(z), \]

(2.19)

then \( F(z) \) starlike with respect to the origin implies \( f(z) \) convex, and conversely. Then (2.17) and (2.3) would be a simple consequence of

\[ * \text{After this paper was completed, the author learned from Professor Z. Nehari that formulae (2.17), (2.18), with a finite number of terms, were given earlier by C. Darwin [2].} \]
the Schwarz-Christoffel transformation

\[ f(z) = c \int_0^t \frac{dt}{\prod_{n=1}^{\infty} (1 - te^{i\theta_n})^{\gamma_n}}. \]  

It is worth noting that even though \( F(z) \) given by (2.17) may map \( E \) onto a region with inaccessible boundary points, the associated \( f(z) \) given by (2.19) and (2.20) maps \( E \) onto a convex region which consequently has no inaccessible boundary points. Thus integration may destroy inaccessible boundary points.

3. Some examples of functions for which \( B_\gamma \) has inaccessible boundary points. In general it may be very difficult to find the zeros of (2.16), but for the problem at hand this is not necessary, since if \( \theta_{n+1} < \theta_n \), and if \( \alpha_n \) is the zero of (2.16) corresponding to this interval, then \( |F(e^{i\alpha_n})| \leq |F(e^{i\theta})| \). This together with Theorem 1 gives the following results.

**Theorem 2.** A necessary and sufficient condition that \( F(z) \) given by (2.3) maps \( E \) onto a region with inaccessible boundary points is that there is a constant \( M \) and a decreasing sequence \( \{ \eta_n \} \) such that

\[ \lim_{n \to \infty} \eta_n = 0, \]

and

\[ |F(e^{i\alpha_n})| \leq M, \quad n = 1, 2, \ldots. \]

Under these conditions all the points \( \omega > M \) will be inaccessible boundary points.

To determine such a sequence we proceed thus. If, in (2.3), \( c = 1 \), then

\[ \log |F(e^{i\gamma})| = \log\frac{1}{2} - \sum_{k=1}^{\infty} \gamma_i \log |\cos \theta - \cos \phi|. \]

Since \( \cos \eta - \cos \theta = -2 \sin (\eta + \theta)/2 \sin (\eta - \theta)/2 \) and \( 2|x|/\pi \leq |\sin x| \leq |x| \) for \( 0 \leq |x| \leq \pi/2 \), it follows that if \( 0 < \eta, \theta \leq \pi/2 \), then

\[ Q(\eta) = \sum_{i=1}^{\infty} \gamma_i \log |\cos \eta - \cos \theta_i|. \]

\[ \geq \log \frac{2}{\pi^2} + \sum_{i=1}^{\infty} \gamma_i \log |\eta - \theta_i|. \]
Let us suppose that, given \( \{ \gamma_n \} \) satisfying (2.5), it is possible to determine an increasing sequence of positive constants \( \{ c_n \} \) such that the series
\[
\sum_{j=1}^{\infty} e^{-cj} = \sigma,
\]
both converge. In \( F(z) \) take \( \theta_n > 0 \) such that
\[
\theta_n^2 = \frac{\pi^2}{4\sigma} \sum_{j=1}^{\infty} e^{-cj}, \quad n = 1, 2, \ldots ,
\]
and note that \( \pi/2 = \theta_1 > \theta_2 > \cdots > \theta_{m}=0 \). Next set
\[
\eta_n = \frac{\theta_n^2 + \theta_{n+1}^2}{2} = \frac{\pi^2}{8\sigma} \left( e^{-\epsilon_0} + 2 \sum_{j=n+1}^{\infty} e^{-cj} \right)
\]
and observe that \( \theta_1 > \eta_1 > \theta_2 > \cdots > \theta_n > \eta_n > \cdots > 0 \). Then for all \( j, n=1, 2, \ldots \),
\[
| \eta_n - \theta_j | \geq \frac{\pi^2}{8\sigma} e^{-cj}.
\]
Using this in (3.4), and taking into account (3.6), gives
\[
Q(\eta_n) > -M_1 - \log 4\sigma.
\]
Consequently from (3.3) and (3.4) we have
\[
| F(e^{i\theta_n}) | < 2\sigma e^{M_1} = M.
\]
We summarize these results as follows:

\textbf{Lemma 1.} Given \( \{ \gamma_n \} \) satisfying (2.5), if it is possible to find an increasing sequence of positive constants \( \{ c_n \} \) such that (3.5) and (3.6) are convergent series, then \( F(z) \) defined by (2.3) with \( \theta_n \) defined by (3.7) maps the unit circle onto a region \( B \), for which all the points \( w > 2\sigma e^{M_1} \) are inaccessible boundary points.

At first the author conjectured that for every convergent series (2.5) there is a sequence of positive constants \( \{ c_n \} \) such that (3.5) and (3.6) both converge. Professor James A. Jenkins refuted this conjecture with the following counter example. Set \( \gamma_n = 1/n(\log n)^2 \), \( n \geq 2 \). If we could find \( c_n > 0 \) such that (3.5) and (3.6) both converge, then the sum
\[
\sum_{j=1}^{\infty} e^{-cj} = \sigma,
\]
would also converge. But this last is impossible, since if \( c_n \geq \log n \), then \( \gamma_n c_n \geq 1/n \log n \), and if \( c_n \leq \log n \), then \( e^{-cn} \geq 1/n \), so that in any case \( e^{-cn} + \gamma_n c_n > 1/n \log n \).

A rather general case in which Lemma 1 can be applied occurs when \( \{\gamma_n\} \) is decreasing and \( \sum \gamma_n \log n \) converges, for then it suffices to take \( c_n = -\log \gamma_n \) so that (3.5) becomes (2.5). This is possible, for example, when \( \gamma_n = kn^{-p}, k > 0, p > 1 \).

A still simpler case occurs when \( \{\gamma_n\} \) is a geometric sequence, i.e. \( \gamma_n = A^{n-1} (1 - A), 0 < A < 1 \). In this case, taking \( c_n = -\log A^{n-1}(1 - A) \), we have, from (3.5), \( \sigma = 1 \), and from (3.6)

\[
M_1 = -\frac{A}{1 - A} \log A - \log(1 - A),
\]

so that

\[
M = \frac{2}{(1 - A)A^{(1/A)/1}}.
\]

Equation (3.7) gives \( \theta_n = \pi A^{(n-1)/2}/2 \). This proves the following theorem.

**Theorem 3.** If \( 0 < A < 1 \), the function

\[
F(z) = \frac{z}{\prod_{n=0}^{\infty} (1 - z^2 \cos (\pi A^{n/2}/2) + z^2)A^{(1-A)}}
\]

maps \( E \) onto a region of type S for which all the points \( w > M \) are inaccessible boundary points, where \( M \) is given by (3.13).

The case \( A = 1/2 \) gives the function (1.1) mentioned in the introduction.

**4. Further examples.** By suitable combinations of \( F(z) \) with certain simple functions, other interesting examples can be obtained. Thus if \( f_c(z) \) maps onto \( E \) onto \( E_n \), then \( \log F(f_c(z)) \) maps \( E \) onto a strip \( \Im(w) < \pi \) cut by lines parallel to the real axis in such a way that the points \( w > \log M \) are inaccessible boundary points. The function \( (F^{1/2}(f_c(z)) - 1)/(F^{1/2}(f_c(z)) + 1) \) maps \( E \) onto the unit circle with circular arc slits so disposed that the points \( (M^{1/2} - 1)/(M^{1/2} + 1) < w \leq 1 \) are inaccessible boundary points. It seems difficult however to give an expression for the function which maps \( E \) onto a circle.
with radial slits and having inaccessible boundary points.

The inaccessible boundary points can be rotated by taking $F_k(z) = e^{i\theta_k}F(ze^{-i\theta_k})$, and it is then clear that

$$\begin{equation}
G_m(z) = \left\{ \prod_{k=1}^{m} F_k(z) \right\}^{1/m}
\end{equation}$$

maps $E$ onto a region having $m$ radial lines with inaccessible boundary points, providing only that the $\theta_n^{(k)}$ for $F_k(z)$ satisfy suitable conditions.

Finally if

$$\begin{equation}
2 > \theta_1 > \theta_2 > \cdots > \theta_{\infty} > \theta^* > 0,
\end{equation}$$

$$\begin{equation}
\gamma^* + \sum_{n=1}^{\infty} \gamma_n = 1, \quad \gamma^*, \gamma_n > 0,
\end{equation}$$

and if the $\theta_n$ are suitably chosen, then

$$\begin{equation}
F(z) = \frac{z}{(1 - z^2 \cos \theta^* + z^2)^{\gamma^*} \prod_{n=1}^{\infty} (1 - z^2 \cos \theta_n + z^2)^{\gamma_n}}
\end{equation}$$

maps $E$ onto a region for which points on the two radial lines $w = \rho e^{\pm i\gamma^*}$, $\rho > M$, are accessible from one side but inaccessible from the other.

References


University of Kentucky