ON TRIGONOMETRIC INTERPOLATION[1]

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It is well known that for every trigonometric polynomial $P(t)$ of degree n and $2n+1$ equidistant points modulo 2π, $t_1, t_2, \cdots, t_{2n+1}$ in the interval $(0, 2\pi)$,

$$(1)\qquad \left\{\int_0^{2\pi} |P(t)|^2 dt\right\}^{1/2} = \left\{\sum_{k=1}^{2n+1} |P(t_k)|^2 \Delta t_k\right\}^{1/2}, \quad \Delta t_k = \frac{2\pi}{2n+1}.$$

In 1936, J. Marcinkiewicz [1][2] extended this relation to exponents other than 2. The above identity is then replaced by two inequalities.

In this note we shall deal with a further extension of (1) concerning the derivatives of trigonometric polynomials. Our starting point will be the inequalities

$$(2)\qquad \begin{aligned} \left\{\sum_{k=1}^{2n+1} \left|\frac{\Delta^r P(t_k)}{(\Delta t)^r}\right|^2 \Delta t_k\right\}^{1/2} &\leqq \left\{\int_0^{2\pi} \left|\frac{d^r P(t)}{dt^r}\right|^2 dt\right\}^{1/2} \\ &\leqq \left(\frac{\pi}{2}\right)^r \left\{\sum_{k=1}^{2n+1} \left|\frac{\Delta^r P(t_k)}{(\Delta t)^r}\right|^2 \Delta t_k\right\}^{1/2} \end{aligned}$$

which in case of derivatives take the place of (1). $(\Delta^r P(t)/(\Delta t)^r$ is the rth divided difference formed with the constant increment $\Delta t = 2\pi/(2n+1))$, and we shall show that here, as in Marcinkiewicz's case, the exponent 2 can be replaced by any $p \geqq 1$. As an application, we shall give bounds for the p-norms of the derivatives of interpolating polynomials in terms of the corresponding quantities for the interpolated function. This, in turn, allows us to derive some information as to how the order of approximation yielded by trigonometric interpolation depends upon the integrability properties of the derivatives of the interpolated function.

To simplify the notation, we shall introduce, in connection with any system of m equidistant points modulo 2π, $t_1, t_2, \cdots, t_m$, the step functions $\omega_m(t)$ defined as follows:

$$\omega_m(t) = t_k \qquad \text{for } t_k \leqq t < t_{k+1}.$$

With this, any sum of the form $\sum_{k=1}^{m} f(t_k)\Delta t_k$ can be written as a Stieltjes integral $\int_0^{2\pi} f(t)d\omega_m$.

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

Moreover, for any function $f(t)$, $f^{(r)}(t)$ and $f^{[r]}(t)$ will designate the rth derivative of f (provided it exists) and the rth divided difference $\Delta^r f(t)/(\Delta t)^r$ of f with regard to a constant increment Δt respectively (the value of Δt will always be clear from the context). We shall make the convention:

$$f^{(0)}(t) = f^{[0]}(t) = f(t).$$

In the first place we shall prove a couple of lemmas which show that the first part of (2), properly extended to any $p \geqq 1$, has a much larger range of functions than the simple trigonometric polynomials.

Lemma 1. *For every periodic function $f(t)$ of period 2π with absolutely continuous derivatives up to the $(r-1)$st order, and any non-negative, nondecreasing convex function $\phi(x)$*

$$\int_0^{2\pi} \phi(|f^{[r]}(t)|)dt \leqq \int_0^{2\pi} \phi(|f^{(r)}(t)|)dt. \tag{3}$$

Proof. By hypothesis

$$\begin{aligned} |f^{[1]}(t)| = \left|\frac{f(t+\Delta t) - f(t)}{\Delta t}\right| &= \frac{1}{\Delta t}\left|\int_t^{t+\Delta t} f^{(1)}(s)ds\right| \\ &\leqq \frac{1}{\Delta t}\int_t^{t+\Delta t} |f^{(1)}(s)|\, ds \end{aligned} \tag{4}$$

and by Jensen inequality (cf. Hardy, Littlewood, and Pólya [2, p. 74])

$$\begin{aligned} \phi(|f^{[1]}(t)|) &\leqq \frac{1}{\Delta t}\int_t^{t+\Delta t} \phi(|f^{(1)}(s)|)ds \\ &= \frac{1}{\Delta t}\int_0^{\Delta t} \phi(|f^{(1)}(s+t)|)ds. \end{aligned} \tag{5}$$

Integrating and exchanging the order of integration on the right, we have

$$\int_0^{2\pi} \phi(|f^{[1]}(t)|)dt \leqq \frac{1}{\Delta t}\int_0^{\Delta t} ds \int_0^{2\pi} \phi(|f^{(1)}(s+t)|)dt \tag{6}$$

which, because of the periodicity of $f^{(1)}(t)$, reduces to (3) with $r=1$. From this, by iteration, follows (3) for any r. Taking $\phi = x^p$ and $\phi = x \log^+ x$,[3] we get:

[3] $\log^+ x$ is defined as the positive part of $\log x$, that is, as $[|\log x| + \log x]/2$.

COROLLARY.

$$\text{(7)}\quad \text{(a)}\quad \left\{\int_0^{2\pi} |f^{[r]}(t)|^p dt\right\}^{1/p} \leqq \left\{\int_0^{2\pi} |f^{(r)}(t)|^p dt\right\}^{1/p}, \ 1 \leqq p \leqq \infty,$$

$$\text{(8)}\quad \text{(b)}\quad \int_0^{2\pi} |f^{[r]}(t)| \log^+ |f^{[r]}(t)|\, dt \leqq \int_0^{2\pi} |f^{(r)}(t)| \log^+ |f^{(r)}(t)|\, dt.$$

LEMMA 2. *For every periodic function $f(t)$ of period 2π with absolutely continuous derivatives up to the $(r-1)$st order and any non-negative, nondecreasing convex function $\phi(x)$*

$$\text{(9)}\qquad \int_0^{2\pi} \phi(|f^{[r]}(t)|)d\omega_m \leqq \int_0^{2\pi} \phi(|f^{(r)}(t)|)dt, \qquad m = 1, 2, \cdots .$$

PROOF. From (5)

$$\text{(10)}\qquad \phi(|f^{[1]}(t_k)|)\Delta t_k \leqq \int_{t_k}^{t_{k+1}} \phi(|f^{(1)}(s)|)ds.$$

Adding with regard to k, we get

$$\text{(11)}\qquad \int_0^{2\pi} \phi(|f^{[1]}(t)|)d\omega_m \leqq \int_0^{2\pi} \phi(|f^{(1)}(s)|)ds$$

which is (9) with $r=1$. To extend the inequality to any r we proceed by induction. Let us assume it to be true for $1, 2, \cdots, r-1$, then

$$\int_0^{2\pi} \phi(|f^{[r]}(t)|)d\omega_m = \int_0^{2\pi} \phi(|f^{[1][r-1]}(t)|\, d\omega_m$$

$$\text{(12)}\qquad \leqq \int_0^{2\pi} \phi(|f^{[1](r-1)}(t)|)dt \leqq \int_0^{2\pi} \phi(|f^{(r-1)[1]}(t)|)dt$$

and by the previous lemma

$$\text{(13)}\qquad \int_0^{2\pi} \phi(|f^{(r-1)[1]}(t)|)dt \leqq \int_0^{2\pi} \phi(|f^{(r)}(t)|)dt,$$

and the inequality holds for r.

As in Lemma 1, we get by specialization of ϕ:

COROLLARY.

$$\text{(14)}\quad \text{(a)}\quad \left\{\int_0^{2\pi} |f^{[r]}(t)|^p d\omega_m\right\}^{1/p} \leqq \left\{\int_0^{2\pi} |f^{(r)}(t)|^p dt\right\}^{1/p},$$

$$1 \leqq p \leqq \infty,$$

$$(15)\ \text{(b)}\quad \int_0^{2\pi} |f^{[r]}(t)| \log^+ |f^{[r]}(t)|\, d\omega_m \leq \int_0^{2\pi} |f^{(r)}(t)| \log^+ |f^{(r)}(t)|\, dt.$$

It is to be noticed that this corollary is no longer true for $r=0$. However, for trigonometric polynomials $P(t)$, its validity can be partially restored, for as Marcinkiewicz has shown [1], if $m \geqq 2n+1$ (n is the degree of P)

$$(15')\qquad \left\{\int_0^{2\pi} |P(t)|^p d\omega_m\right\}^{1/p} \leqq A \left\{\int_0^{2\pi} |P(t)|^p dt\right\}^{1/p}$$

where A is an absolute constant greater than one.

We now proceed to the generalization of the second part of (2). Here, without exception, we shall have to restrict our consideration to trigonometric polynomials.

THEOREM 1. *For any trigonometric polynomial $P(t)$ of degree n, and $\Delta t \leqq 2\pi/(2n+1)$,*[4]

$$(16)\ \text{(a)}\quad \left\{\int_0^{2\pi} |P^{(r)}(t)|^p dt\right\}^{1/p} \leqq A_{p,r} \left\{\int_0^{2\pi} |P^{[r]}(t)|^p dt\right\}^{1/p}, \qquad 1 < p < \infty,$$

$$(17)\ \text{(b)}\quad \int_0^{2\pi} |P^{(r)}(t)|\, dt \leqq A_{1,r} \int_0^{2\pi} |P^{[r]}(t)| \log^+ |P^{[r]}(t)|\, dt + C_r,$$

(c) *if* $|P^{[r]}(t)| \leqq 1$, *then*

$$(18)\qquad \int_0^{2\pi} e^{\lambda |P^{(r)}(t)|} dt \leqq \nu_{\lambda,r} \qquad \text{for } 0 \leqq \lambda < \Lambda_r$$

where the quantities $A_{p,r}$, C_r, $\nu_{\lambda,r}$, Λ_r are all finite and depend on their subindices only.

PROOF. If $P(t) = \sum_{j=-n}^{n} c_j e^{ijt}$, by a simple computation one gets

$$(19)\qquad P^{(r)}(t) = \sum_{j=-n}^{n} c_j (ij)^r e^{ijt},$$

$$(20)\qquad P^{[r]}(t) = \sum_{j=-n}^{n} c_j (ij)^r \left(\frac{\sin j(\Delta t/2)}{j\Delta t/2}\right)^r e^{ij(t+r\Delta t/2)}$$

[4] This theorem is still valid for $\Delta t < \pi/2n$ provided that the constants are made to depend on the value of Δt.

so

$$P^{[r]}\left(t - r\frac{\Delta t}{2}\right) = \sum_{j=-n}^{n} c_j(ij)^r \left(\frac{\sin jt(\Delta t/2)}{j\Delta t/2}\right)^r e^{ijt}. \tag{21}$$

Let $s_j(t)$ be the jth partial sum of the Fourier series of $P^{[r]}(t-r\Delta t/2)$; then

$$s_j(t) - s_{j-1}(t) = \left[c_j(ij)^r e^{ijt} + c_{-j}(-ij)^r e^{-ijt}\right]\left(\frac{\sin j\Delta t/2}{j\Delta t/2}\right)^r \tag{22}$$

which inserted in the development of $P^{(r)}(t)$ gives

$$\begin{aligned} P^{(r)}(t) &= \sum_{j=1}^{n}\left(\frac{j\Delta t/2}{\sin j\Delta t/2}\right)^r \left[s_j(t) - s_{j-1}(t)\right] \\ &= \sum_{j=1}^{n-1}\left[\left(\frac{j\Delta t/2}{\sin j\Delta t/2}\right)^r - \left(\frac{(j+1)\Delta t/2}{\sin (j+1)\Delta t/2}\right)^r\right] s_j(t) \\ &\quad + \left(\frac{n\Delta t/2}{\sin n\Delta t/2}\right)^r s_n(t). \end{aligned} \tag{23}$$

Taking p-norms in both sides and applying Minkowski inequality to the right-hand member, we get

$$\begin{aligned} \left\{\int_0^{2\pi} \left| P^{(r)}(t)\right|^p dt\right\}^{1/p} \leqq \sum_{j=1}^{n-1} &\left| \left(\frac{j\Delta t/2}{\sin j\Delta t/2}\right)^r \right. \\ &\left. - \left(\frac{(j+1)(\Delta t/2)}{\sin (j+1)(\Delta t/2)}\right)^r \right| \left\{\int_0^{2\pi} \left| s_j(t)\right|^p dt\right\}^{1/p} \\ &+ \left(\frac{n\Delta t/2}{\sin n\Delta t/2}\right)^r \left\{\int_0^{2\pi} \left| s_n(t)\right|^p dt\right\}^{1/p}. \end{aligned} \tag{24}$$

If $1 < p < \infty$, the Riesz inequality [3] relating the p-norm of a function with the p-norms of the partial sums of its Fourier series yields

$$\left\{\int_0^{2\pi} \left| s_j(t)\right|^p dt\right\}^{1/p} \leqq A_p \left\{\int_0^{2\pi} \left| P^{[r]}(t - r(\Delta t/2))\right|^p dt\right\}^{1/p}$$

where A_p depends on p only. For $j = n$ one can take $A_p = 1$ since

$$s_n(t) = P^{[r]}(t - r\Delta t/2).$$

Inserting this in (24) and taking account of the periodicity of $P^{[r]}(t)$ and of the fact that the function $x/\sin x$ increases from 1 to $\pi/2$ when t goes from 0 to $\pi/2$, one obtains

$$(25)\qquad \left\{\int_0^{2\pi} | P^{(r)}(t) |^p dt\right\}^{1/p} \leqq \left[\left(\left(\frac{\pi}{2}\right)^r - 1\right)A_p + \left(\frac{\pi}{2}\right)^r\right]\left\{\int_0^{2\pi} | P^{[r]}(t) |^p dt\right\}^{1/p}$$

which is (16) with $A_{p,r} = ((\pi/2)^r - 1)A_p + (\pi/2)^r$.

If $p = 1$, one proceeds similarly from (24), but instead of using Riesz's theorem one uses the inequality

$$(26)\qquad \int_0^{2\pi} | s_j(t) | \, dt \leqq A_1 \int_0^{2\pi} | P^{[r]}(t) | \log^+ | P^{[r]}(t) | \, dt + C_1$$

where A_1 and C_1 are absolute constants. (Cf. Zygmund [**4**, p. **154**].) One gets

$$(27)\qquad \begin{aligned} &\int_0^{2\pi} | P^{(r)}(t) | \, dt \\ &\leqq \left[\left(\frac{\pi}{2}\right)^r - 1\right]\left[A_1 \int_0^{2\pi} | P^{[r]}(t) | \log^+ | P^{[r]}(t) | \, dt + C_1\right] \\ &\quad + \left(\frac{\pi}{2}\right)^r \int_0^{2\pi} | P^{[r]}(t) | \, dt \end{aligned}$$

which obviously implies (17).

Finally, to prove (18), we integrate the inequality

$$(28)\qquad e^{\lambda | P^{(r)}(t) |} \leqq 2 \cosh \lambda | P^{(r)}(t) | = 2 \sum_{p=0}^{\infty} \frac{\lambda^{2p}}{(2p)!} | P^{(r)}(t) |^p$$

and majorate

$$\int_0^{2\pi} | P^{(r)}(t) |^{2p} dt$$

with the help of (16),

$$(29)\qquad \int_0^{2\pi} e^{\lambda | P^{(r)}(t) |} dt \leqq 2 \sum_{p=0}^{\infty} \frac{\lambda^{2p}}{(2p)!} (A_{p,r})^{2p} \int_0^{2\pi} | P^{[r]}(t) |^{2p} dt.$$

Since $A_p = O(p)$ (cf. A. Zygmund [**4**, p. 149]) the power series $\sum_{p=0}^{\infty} (\lambda^{2p}/(2p)!)(A_{p,r})^{2p}$ has positive radius of convergence and the proof is complete.

THEOREM 2. *For any trigonometric polynomial $P(t)$ of degree n*

$$(30)\quad \text{(a)}\quad \left\{\int_0^{2\pi} |P^{(r)}(t)|^p dt\right\}^{1/p} \leqq B_{p,r}\left\{\int_0^{2\pi} |P^{[r]}(t)|^p d\omega_{2n+1}, \qquad 1 < p < \infty,\right.$$

$$(31)\quad \text{(b)}\quad \int_0^{2\pi} |P^{(r)}(t)|\, dt \leqq B_{1,r}\int_0^{2\pi} |P^{[r]}(t)| \log^+ |P^{[r]}(t)|\, d\omega_{2n+1} + D_r,$$

(c) *if* $|P^{[r]}(t_k)| \leqq 1$, $k=1, \cdots, 2n+1$, *then*

$$(32)\qquad \int_0^{2\pi} e^{\lambda|P^{(r)}(t)|^{1/2}}dt \leqq \mu_{\lambda,r} \qquad \textit{for } 0 \leqq \lambda < \lambda_r$$

where the quantities $B_{p,r}$, D_r, $\mu_{\lambda,r}$, λ_r *are all finite and depend on their subindices only.*

Proof. For $r=0$, this has been proved by J. Marcinkiewicz [1]. For $r>0$, (a) follows immediately from Theorem 1 (a) in conjunction with Marcinkiewicz's result applied to the trigonometric polynomial $P^{[r]}(t)$. In this way one obtains $B_{p,r}=B_{p,0}A_{p,r}$. This reasoning does not work in (b) because $|P^{[r]}(t) \log^+ |P^{[r]}(t)|$ is no longer a trigonometric polynomial. However, to prove it one can apply an argument used by J. Marcinkiewicz and A. Zygmund [5] in establishing the analogous inequality for $r=0$.[5] Here is how we proceed. Let $g(t)=\exp(i \arg P^{(r)}(t))$ and designate $g_n(t)=\sum_{j=-n}^{n} d_j e^{ijt}$ the nth partial sum of the Fourier series of $g(t)$, then

$$(33)\qquad \int_0^{2\pi} |P^{(r)}(t)|\, dt = \int_0^{2\pi} P^{(r)}(t)\bar{g}(t)dt = \int_0^{2\pi} P^{(r)}(t)\bar{g}_n(t)dt.$$

Now $P^{(r)}(t)=\sum_{j=-n}^{n} c_j(ij)^r e^{ijt}$ and the above is equal to $2\pi\sum_{j=-n}^{n} c_j(ij)^r\bar{d}_j$ which in turn is equal to the scalar product of $P^{[r]}(t-r\Delta t/2)$ (see (21)) and $g_n^*(t)=\sum_{j=-n}^{n} d_j(j\Delta t/2/\sin j\Delta t/2)^r e^{ijt}$. Thus

$$(34)\qquad \begin{aligned}\int_0^{2\pi} |P^{(r)}(t)|\, dt &= \int_0^{2\pi} P^{[r]}(t - r(\Delta t/2)\bar{g}_n^*(t)dt \\ &= \int_0^{2\pi} P^{[r]}(t)\bar{g}_n^*(t + r(\Delta t/2))dt.\end{aligned}$$

The last terms being the integral of the product of two trigonometric polynomials of degree n, we can, as a consequence of (1), replace dt

[5] This was kindly communicated to the author by Professor A. Zygmund.

by $d\omega_{2n+1}$ so

$$\int_0^{2\pi} |P^{(r)}(t)|\,dt = \int_0^{2\pi} P^{[r]}(t)\bar{g}_n^*(t + r(\Delta t/2))d\omega_{2n+1}. \tag{35}$$

Next we apply Young's inequality: $\mu\nu \leqq \mu \log^+ \mu + e^\nu$, $\mu, \nu \geqq 0$ (cf. Hardy, Littlewood, and Pólya [2, p. 61, Exc. 62]) to the product under the sum (35) and obtain

$$\begin{aligned}\int_0^{2\pi} |P^{(r)}(t)|\,dt \leqq \int_0^{2\pi} \left|\frac{P^{[r]}(t)}{\alpha}\right| \log^+ \left|\frac{P^{[r]}(t)}{\alpha}\right| d\omega_{2n+1} \\ + \int_0^{2\pi} \exp\left(\alpha\left| g_n^*(t + 2\Delta t/2)\right|\right) d\omega_{2n+1}\end{aligned} \tag{36}$$

where α is a positive constant to be determined later. To complete the proof it is enough to show that there is an α such that the last integral on the right is bounded and independent of P. For that purpose we notice, in the first place, that $d\omega_{2n+1}$ can be essentially replaced by dt. In fact, using the inequalities $e^u \leqq 2\sum_{p=0}^{\infty} u^{2p}/(2p)! \leqq e^u + 1$ and (15′), we obtain

$$\begin{aligned}&\int_0^{2\pi} e^{\alpha|g_n^*(t+r\Delta t/2)|}d\omega_{2n+1} \\ &\leqq 2\sum_{p=0}^{\infty} \frac{\alpha^{2p}}{(2p)!}\int_0^{2\pi} |g_n^*(t + r\Delta t/2)|^{2p} d\omega_{2n+1} \qquad (37)\\ &\leqq 2\sum_{p=0}^{\infty} \frac{(\alpha A)^{2p}}{(2p)!}\int_0^{2\pi} |g_n^*(t + r\Delta t/2)|^{2p} dt \\ &\leqq \int_0^{2\pi} (e^{\alpha A|g_n^*(t+r\Delta t/2)|} + 1)dt = \int_0^{2\pi} e^{\alpha A|g_n^*(t)|}dt + 2\pi. \qquad (38)\end{aligned}$$

Now from the definition of $g_n^*(t)$ we derive by summation by parts, as in Theorem 1,

$$g_n^*(t) = \sum_{j=0}^{n} \beta_j g_j(t) \tag{39}$$

where

$$\beta_j = \left(\frac{j\Delta t/2}{\sin j\Delta t/2}\right)^r - \left(\frac{(j+1)\Delta t/2}{\sin (j+1)\Delta t/2}\right)^r$$

$$\text{for } j = 0, 1, \cdots, n-1 \quad \text{and} \quad \beta_n = \left(\frac{n\Delta t/2}{\sin n\Delta t/2}\right)^r.$$

By the monotonicity and convexity of the exponential, it follows that

$$\int_0^{2\pi} \exp\,(\alpha A \mid \overset{*}{g}_n(t) \mid)dt \tag{40}$$

$$\leqq \frac{\sum_{j=0}^{n} \mid \beta_j \mid \int_0^{2\pi} \exp\left((\alpha A \sum_{k=0}^{n} \mid \beta_k \mid) \mid g_j(t) \mid \right) dt}{\sum_{k=0}^{n} \mid \beta_k \mid}.$$

But

$$\sum_{k=0}^{n} \mid \beta_k \mid = 2\beta_n - 1 \leqq 2(\pi/2)^r - 1,$$

so, if we take $\alpha = 1/A\,[2(\pi/2)^r - 1]$, (40) yields

$$\int_0^{2\pi} e^{\alpha A \mid \overset{*}{g}_n(t) \mid} dt \leqq \frac{\sum_{j=0}^{n} \mid \beta_j \mid \int_0^{2\pi} e^{\mid g_j(t) \mid} dt}{\sum_{k=0}^{n} \mid \beta_k \mid}. \tag{41}$$

The $g_k(t)$ being the partial sums of the Fourier series of a function $g(t)$, with $|g(t)| \leqq 1$, there is, according to a lemma of A. Zygmund [4, p. 164], an absolute constant γ such that for every j, $\int_0^{2\pi} e^{|g_j(t)|} dt < \gamma$. Hence, inserting in (41) and going back to (37), we obtain

$$\int_0^{2\pi} e^{\alpha \mid \overset{*}{g}_j(t + r\Delta t/2) \mid} d\omega_{2n+1} \leqq \gamma + 2\pi \tag{42}$$

which proves (b).

Finally one proves (c) as in Theorem 1, from the inequality

$$\int_0^{2\pi} e^{\lambda \mid P^{(r)}(t) \mid^{1/2}} dt \leqq 4\pi \sum_{p=0}^{\infty} \frac{\lambda^{2p}}{(2p)!} (B_{2p,r})^p, \quad \mid P^{[r]}(t_k) \mid \leqq 1, \tag{43}$$

by observing that since $B_{p,0} = O(p)$ (cf. J. Marcinkiewicz and A. Zygmund [5, p. 140]) and $A_{p,r} = O(p)$, the power series on the right has a positive radius of convergence.

We now pass to the announced applications to trigonometric interpolation.

THEOREM 3. *Let $f(t)$ be a periodic function of period 2π with absolutely continuous derivatives up to the $(r-1)$st order and let $P(t)$ be a trigono-*

metric polynomial of degree n coinciding with f(t) at more than 2n equidistant points; then

$$(44)\quad (a)\quad \left\{\int_0^{2\pi} |P^{(r)}(t)|^p dt\right\}^{1/p} \leqq B_{p,r}\left\{\int_0^{2\pi} |f^{(r)}(t)|^p dt\right\}^{1/p},$$

$$(45)\quad (b)\quad \int_0^{2\pi} |P^{(r)}(t)|\, dt \leqq B_{1,r}\int_0^{2\pi} |f^{(r)}(t)| \log^+ |f^{(r)}(t)|\, dt.$$

(c) *If* $|f^{(r)}(t)| \leqq 1$,

$$(46)\qquad \int_0^{2\pi} e^{\lambda|P^{(r)}(t)|^{1/2}} dt \leqq \mu_{\lambda,r} \qquad \text{for } 0 \leqq \lambda < \lambda_r,$$

where the constants $B_{p,r}$, D_r, $\mu_{\lambda,r}$, λ_r are the same as in Theorem 2.

Proof. (a) and (b) follow immediately from the identities

$$(47)\quad \left\{\int_0^{2\pi} |P^{[r]}(t)|^p d\omega_{2n+1}\right\}^{1/p} = \left\{\int_0^{2\pi} |f^{[r]}(t)|^p d\omega_{2n+1}\right\}^{1/p}, \qquad 1 < p < \infty,$$

$$(48)\quad \int_0^{2\pi} |P^{[r]}(t)| \log^+ |P^{[r]}(t)|\, d\omega_{2n+1} = \int_0^{2\pi} |f^{[r]}(t)| \log^+ |f^{[r]}(t)|\, d\omega_{2n+1}$$

in conjunction with Theorem 2 (a) and (b) and the corollary to Lemma 2. As to (c), it follows from Theorem 2 (c) and the fact that $|f^{(r)}(t)| \leqq 1$ implies $|f^{[r]}(t_k)| \leqq 1$.

Theorem 4. *Let f(t) be a periodic function of period 2π with absolutely continuous derivatives up to the $(r-1)$st order and let P(t) be a trigonometric polynomial of degree n coinciding with f(t) in more than 2n equidistant points modulo 2π; then*

$$(49)\quad (a)\quad |f(t) - P(t)| \leqq \left(\frac{1}{n}\right)^{r-1/p} M_{r,p}\left\{\int_0^{2\pi} |f^{(r)}(t)|^p dt\right\}^{1/p}, \qquad 1 < p < \infty,$$

$$(50)\quad (b)\quad \left\{\int_0^{2\pi} |f(t) - P(t)|^p dt\right\}^{1/p} \leqq \left(\frac{1}{n}\right)^{r} N_{r,p}\left\{\int_0^{2\pi} |f^{(r)}(t)|^p dt\right\}^{1/p}, \qquad 1 < p < \infty,$$

$$(51)\quad (c)\quad \int_0^{2\pi} |f(t) - P(t)|\,dt \leqq \left(\frac{1}{n}\right)^r \left\{N_{r,1}\int_0^{2\pi} |f^{(r)}(t)| \log^+ |f^{(r)}(t)|\,dt + Q_r\right\}$$

where $M_{r,p}$, $N_{r,p}$, and Q_r are constants depending on their subindices only.

PROOF. Let $h(x)$ be a function vanishing at $x=0$, $x=\pi$, and having in that interval absolutely continuous derivatives up to the $(r-1)$st order and an absolutely integrable rth derivative. Under these conditions $h(x)$, extended as an odd function to the interval $(-\pi, \pi)$, has a uniformly convergent sine Fourier series

$$h(x) = \sum_{k=1}^{\infty} a_k \sin kx \tag{52}$$

from which, by formal differentiation, the Fourier series of the successive derivatives are obtained. Thus

$$h^{(2s)}(x) \sim (-1)^s \sum_{k=1}^{\infty} (k)^{2s} a_k \sin kx, \tag{53}$$

$$h^{(2s+1)}(x) \sim (-1)^s \sum_{k=1}^{\infty} (k)^{2s+1} a_k \cos kx. \tag{54}$$

Therefore

$$a_k = \begin{cases} (-1)^s \dfrac{2}{\pi}\displaystyle\int_0^{2\pi} h^{(2s)}(y)\, \frac{\sin ky}{k^{2s}}\,dy, \\ (-1)^s \dfrac{2}{\pi}\displaystyle\int_0^{2\pi} h^{(2s+1)}(y)\, \frac{\cos ky}{k^{2s+1}}\,dy. \end{cases} \tag{55}$$

Replacing these values of the Fourier coefficients in (52) and interchanging the sum with the integral (permitted in this case), one obtains the following integral representations of $h(x)$

$$h(x) = \int_0^{\pi} h^{(r)}(y) K_r(x, y)\,dy \tag{56}$$

where

$$K_r(x, y) = \begin{cases} (-1)^{r/2} \dfrac{2}{\pi} \displaystyle\sum_{k=1}^{\infty} \frac{\sin kx \sin ky}{k^r} & \text{for } r \text{ even}, \\ (-1)^{(r-1)/2} \dfrac{2}{\pi} \displaystyle\sum_{k=1}^{\infty} \frac{\sin kx \cos ky}{k^r} & \text{for } r \text{ odd}. \end{cases} \tag{57}$$

Except for $K_1(x, y)$, all $K_r(x, y)$ are continuous functions of (x, y) in the square $(0 \leqq x \leqq \pi,\ 0 \leqq y \leqq \pi)$. As to $K_1(x, y)$ it is

$$K_1(x, y) = \begin{cases} 1 - x/\pi, & 0 \leqq y < x, \\ 1/2 - x/\pi, & y = x, \\ -x/\pi, & x < y \leqq \pi. \end{cases} \tag{58}$$

Now by Hölder inequality applied to (56)

$$| h(x) | \leqq \left\{ \int_0^\pi | h^r(y) |^p dy \right\}^{1/p} \left\{ \int_0^\pi | K_r(x, y) |^q dy \right\}^{1/q} \tag{59}$$

and raising this to the pth power and integrating, we obtain

$$\begin{aligned} &\left\{ \int_0^\pi | h(x) |^p dx \right\}^{1/p} \\ &\leqq \left\{ \int_0^\pi | h^{(r)}(y) |^p dy \right\}^{1/p} \left\{ \int_0^\pi \left\{ \int_0^\pi | K_r(x, y) |^q dy \right\}^{p/q} dx \right\}^{1/p}. \end{aligned} \tag{60}$$

Through the substitution

$$x = \pi(t - a)/(b - a)$$

we derive for a function $g(t) = h(\pi(t-a)/(b-a))$ vanishing at the end points of the interval (a, b)

$$\begin{aligned} | g(t) | \leqq &\left(\frac{b - a}{\pi} \right)^{r-1/p} \\ &\cdot \left\{ \int_a^b | g^{(r)}(t) |^p dt \right\}^{1/p} \sup_{0 \leqq x \leqq \pi} \left\{ \int_0^\pi | K_r(x, y) |^q dy \right\}^{1/q}, \end{aligned} \tag{61}$$

$$\begin{aligned} &\left\{ \int_a^b | g(t) |^p dt \right\}^{1/p} \leqq \left(\frac{b - a}{\pi} \right)^r \\ &\cdot \left\{ \int_a^b | g^{(r)}(t) |^p dt \right\}^{1/p} \left\{ \int_0^\pi \left\{ \int_0^\pi | K_r(x, y) |^q dy \right\}^{p/q} dx \right\}^{1/p}. \end{aligned} \tag{62}$$

If in addition $g(t)$ is periodic of period 2π and vanishes at $2n+1$ equidistant points modulo 2π,

$$\begin{aligned} &| g(t) | \leqq \left(\frac{1}{n + 1/2} \right)^{r-1/p} \\ &\cdot \left\{ \int_0^{2\pi} | g^{(r)}(t) |^p dt \right\}^{1/p} \sup_{0 \leqq x \leqq \pi} \left\{ \int_0^\pi | K_r(x, y) |^q dy \right\}^{1/q}, \end{aligned} \tag{63}$$

$$
(64)\quad \begin{aligned} \left\{\int_0^{2\pi} |g(t)|^p dt\right\}^{1/p} &\leqq \left(\frac{1}{n+1/2}\right)^r \\ &\cdot \left\{\int_0^{2\pi} |g^{(r)}(t)|^p dt\right\}^{1/p} \left\{\int_0^{\pi}\left\{\int_0^{\pi} |K_r(x,y)|^q dy\right\}^{p/q} dx\right\}^{1/p}. \end{aligned}
$$

The proof is concluded by setting $g(t)=f(t)-P(t)$ and observing that by Minkowski inequality and the previous theorem

$$
(65)\quad \begin{aligned} &\left\{\int_0^{2\pi} |f^{(r)}(t) - P^{(r)}(t)|^p dt\right\}^{1/p} \\ &\qquad \leqq (1 + B_{p,r})\left\{\int_0^{2\pi} |f^{(r)}(t)|^p dt\right\}^{1/p}, \end{aligned}
$$

$$
(66)\quad \begin{aligned} &\int_0^{2\pi} |f^{(r)}(t) - P^{(r)}(t)|\, dt \\ &\qquad \leqq \int_0^{2\pi} |f^{(r)}(t)|\, dt + A_{1,r}\int_0^{2\pi} |f^{(r)}(t)| \log^+ |f^{(r)}(t)|\, dt + D_r. \end{aligned}
$$

Remark. In the above treatment we have left aside all questions concerning exponents p between 0 and 1 as well as those involving conjugate functions. The interested reader will find no difficulty in extending our results to those cases by following the patterns set here and in the paper of J. Marcinkiewicz and A. Zygmund [5].

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