A GENERALIZATION OF A THEOREM BY HARDY AND LITTLEWOOD

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1. Suppose that $\phi(x)$ is non-negative and integrable in (0, 1), so that it is measurable and finite almost everywhere. If $M(y)$ is the measure of the set in which $\phi(x) \geqq y$, $M(y)$ is a decreasing function of y. The inverse $\bar{\phi}$ of M is defined by

$$\bar{\phi}\{M(y)\} = y,$$

and $\bar{\phi}(x)$ is a decreasing function of x defined uniquely in (0, 1) except for at most an enumerable set of values of x, viz., those corresponding to intervals of constancy of $M(y)$. We may complete the definition of $\bar{\phi}(x)$ by agreeing, for example, that

$$\bar{\phi}(x) = \{\bar{\phi}(x-0) + \bar{\phi}(x+0)\}/2$$

at a point of discontinuity.

We call $\bar{\phi}(x)$ the rearrangement of $\phi(x)$ in decreasing order.

2. The following theorem which is important for its function-theoretic applications is due to Hardy and Littlewood [1]. The theorem may be stated in two equivalent forms.

THEOREM A. *Suppose that $f(x)$ is non-negative and integrable in* (0, 1), *that*

$$\Theta(x) = \Theta(x, f) = \max_{0 \leqq \xi < x} \frac{1}{x - \xi} \int_{\xi}^{x} f(t) dt.$$

Then

$$\bar{\Theta}(x) \leqq \frac{1}{x} \int_{0}^{x} \bar{f}(t) dt$$

for $0 < x \leqq 1$.

THEOREM B. *Suppose that $f(x)$ satisfies the conditions of Theorem* A, *and that $s(y)$ is any increasing function of y defined for $y \geqq 0$. Then*

$$\int_{0}^{1} s\{\Theta(x)\} dx \leqq \int_{0}^{1} s\left\{\frac{1}{x} \int_{0}^{x} \bar{f}(t) dt\right\} dx.$$

In this note I generalize the definition of $\Theta(x)$, and prove theorems analogous to those above.

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3. Suppose that $k(t)$ is a non-negative, decreasing function of t, defined and integrable in (0, 1).

THEOREM 1. *Suppose that* $f(x)$ *is non-negative and integrable in* (0, 1), *that* $k(t)$ *satisfies the conditions of the last paragraph, that* $k(t)\bar{f}(t)$ *is integrable in* (0, 1), *and that*

$$\Theta(x) = \Theta(x, f) = \max_{0<h\leqq x} \frac{1}{h}\int_0^h k\left(\frac{t}{h}\right)f(x-h+t)dt. \tag{1}$$

Then

$$\bar{\Theta}(x) \leqq \frac{1}{x}\int_0^x k\left(\frac{t}{x}\right)\bar{f}(t)dt$$

for $0<x\leqq 1$.

THEOREM 2. *Suppose that* $f(x)$ *satisfies the conditions of Theorem* 1, *and that* $s(y)$ *is any increasing function of* y *defined for* $y\geqq 0$. *Then*

$$\int_0^1 s\{\Theta(x)\}dx \leqq \int_0^1 s\left\{\frac{1}{x}\int_0^x k\left(\frac{t}{x}\right)\bar{f}(t)dt\right\}dx.$$

The equivalence of Theorems 1 and 2 may be proved in the same way as is used to prove the equivalence of Theorems A and B [2, 10.18].

It is easily seen that Theorems A and B are the particular case of Theorems 1 and 2 where $k(t)$ is the kernel of the first Cesaro mean, $(C, 1)$. The latter theorems will also deal with the case (C, δ), $0<\delta\leqq 1$.

In this paper I prove Theorem 2, from which Theorem 1 may be deduced. Firstly, we shall prove Theorem 2 for the special case where

$$s(t) = s_\alpha(t) = \begin{cases} 0 & \text{for} \quad 0 \leqq t \leqq \alpha, \\ 1 & \text{for} \quad t > \alpha. \end{cases} \tag{2}$$

In this case, if we rewrite Theorem 2, it is required to prove that

$$\underset{0\leqq x\leqq 1}{mE}\ [x, \Theta(x) > \alpha] \leqq \underset{0\leqq x\leqq 1}{mE}\left[x, \frac{1}{x}\int_0^x k\left(\frac{t}{x}\right)\bar{f}(t)dt > \alpha\right]. \tag{3}$$

We require the following lemmas:

LEMMA 1. *Suppose that each point of a set in* (a, b) *is the right-hand end point of one or more intervals* $(x-h_x, x)$ *of a family* H. *Then there is a finite nonoverlapping set* S *of intervals of* H *which includes a subset* E_ϵ' *of* E *such that* $mE_\epsilon' > mE-\epsilon$.

This result is due to Sierpinski [3].

LEMMA 2. *If $f(x)$ is non-negative and integrable in* $(0, a+b)$, $k(t)$ *is a non-negative, decreasing function of t in* $(0, 1)$, $k(t)\bar{f}(t)$ *is integrable, and if*

$$(4)\qquad \frac{1}{a}\int_0^a k\left(\frac{t}{a}\right)f(t)dt > \alpha, \qquad \frac{1}{b}\int_0^b k\left(\frac{t}{b}\right)f(t+a)dt > \alpha,$$

then

$$\frac{1}{a+b}\int_0^{a+b} k\left(\frac{t}{a+b}\right)\bar{f}(t)dt > \alpha.$$

Since $k(t)$ is decreasing and integrable in $(0, 1)$, it has a non-negative derivative almost everywhere. Also, writing

$$F(x) = \int_0^x f(t)dt,$$

and from (4), integrating by parts, we have

$$k(1)F(a) + \int_0^1 k'(t)F(at)dt > a\alpha,$$

$$k(1)\{F(a+b) - F(a)\} + \int_0^1 k'(t)\{F(a+bt) - F(a)\}dt > b\alpha.$$

Adding, we have

$$(5)\quad k(1)F(a+b) + \int_0^1 k'(t)\{F(at) + F(a+bt) - F(a)\}dt > (a+b)\alpha.$$

Similarly, we have

$$\int_0^{a+b} k\left(\frac{t}{a+b}\right)\bar{f}(t)dt$$

$$= k(1)\int_0^{a+b}\bar{f}(t)dt + \int_0^1 k'(t)\left\{\int_0^{(a+b)t}\bar{f}(s)ds\right\}dt$$

$$= k(1)F(a+b) + \int_0^1 k'(t)\left\{\int_0^{(a+b)t}\bar{f}(s)ds\right\}dt.$$

Now it is evident from the definition of $\bar{f}(t)$ that, when $mE = (a+b)t$,

$$\int_0^{(a+b)t}\bar{f}(s)ds \geqq \int_E f(s)ds.$$

Hence,

$$\int_0^{(a+b)t} \bar{f}(s)ds \geqq F(at) + F(a + bt) - F(a),$$

and, since almost everywhere $k'(t) \geqq 0$, we have

$$\int_0^{a+b} k\left(\frac{t}{a+b}\right)\bar{f}(t)dt \geqq k(1)F(a+b)$$

$$+ \int_0^1 k'(t)\{F(at) + F(a+bt) - F(a)\}dt > (a+b)\alpha,$$

from which the result follows.

4. We are now in a position to prove (3).

From the definition of $\Theta(x)$ it is seen that to every point x of $E_{0 \leqq x \leqq 1}[x, \Theta(x) > \alpha]$ there exists at least one interval $(x - h_x, x) \subset (0, 1)$ for which

$$\frac{1}{h_x}\int_0^{h_x} k\left(\frac{t}{h_x}\right)f(x - h_x + t)dt > \alpha.$$

Let us apply Lemma 1 to the set E. Then there exists a finite set S of the above intervals which are nonoverlapping, which cover a subset E'_ϵ of E such that $mE'_\epsilon > mE - \epsilon$. It is evidently sufficient to prove (3) when we replace mE in the left-hand side by mE'_ϵ for all $\epsilon > 0$, and, a fortiori, it is sufficient to prove that

$$mS \leqq \underset{0 \leqq x \leqq 1}{mE}\left[x, \frac{1}{x}\int_0^x k\left(\frac{t}{x}\right)\bar{f}(t)dt > \alpha\right]. \tag{6}$$

We further use the legitimate simplification that $f(t) = 0$ for t in the complement of S, since this decreases the right-hand side of (6). This latter assumption permits us to translate the intervals of S to the left until they are end to end, and completely cover the interval $(0, mS)$, i.e., we have the n intervals (j_r, j_{r+1}) with $0 = j_0 \leqq j_r < j_{r+1} \leqq j_n = mS$.

Consider the two intervals (j_{n-2}, j_{n-1}), (j_{n-1}, j_n).

We have

$$\frac{1}{j_n - j_{n-1}}\int_0^{j_n - j_{n-1}} k\left(\frac{t}{j_n - j_{n-1}}\right)f(t)dt > \alpha,$$

$$\frac{1}{j_{n-1} - j_{n-2}}\int_0^{j_{n-1} - j_{n-2}} k\left(\frac{t}{j_{n-1} - j_{n-2}}\right)f(t)dt > \alpha.$$

Let $f_1(t)$ be the decreasing rearrangement of $f(t)$ within the above two intervals. Then, applying Lemma 2, we have

$$\frac{1}{j_n - j_{n-2}} \int_0^{j_n - j_{n-2}} k\left(\frac{t}{j_n - j_{n-2}}\right) f_1(t)dt > \alpha.$$

We have thus reduced the case from n intervals to $(n-1)$ intervals. Repeating the process a further $(n-2)$ times we arrive at $\bar{f}(t)$ defined in the single interval $(0, mS)$, and have

$$\frac{1}{mS} \int_0^{mS} k\left(\frac{t}{mS}\right) \bar{f}(t)dt > \alpha$$

which proves (6).

Thus we have proved the theorem for the special case (2).

5. Since $s(t)$ is an increasing function of t, we may approximate to it uniformly in any finite interval $(0, N)$ by the sum of a finite number of functions of the type $s_\alpha(t)$. The deduction of Theorem 2 from the special case considered above is apparent.

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