MINIMAX SOLUTIONS OF ORDINARY DIFFERENTIAL SYSTEMS

L. P. BURTON AND WILLIAM M. WHYBURN

1. **Introduction.** Let $f_i(x, y_1, \cdots, y_n)$, $i=1, \cdots, n$, be defined and continuous over a region R: $(x, y_1, \cdots, y_n)$ of $(n+1)$-dimensional space. Let P: $(x_0, y_{10}, \cdots, y_{n0})$ be a point of R and consider the differential system

$$\text{(S)} \qquad y_i' = f_i(x, y_1, \cdots, y_n), \qquad i = 1, \cdots, n.$$

The classical existence theorem of Peano states that there is at least one solution[1] $y_1(x), \cdots, y_n(x)$ of (S) through P existing in[2] R over some interval $x_0-h \leqq x \leqq x_0+k$, where h, $k>0$. If for some such interval this solution through P is not unique, then there are infinitely many solutions through P and the existence of critical type solutions is a possibility. W. Osgood,[3] P. Montel,[4] and O. Perron,[5] using different methods, considered the case $n=1$ and proved the existence of a maximum and a minimum solution. E. Kamke[6] gave an example to show that for $n>1$ there will not in general be a maximum and a minimum solution through P, but that such solutions do exist provided $f_i(x, y_1, \cdots, y_n)$, $i=1, \cdots, n$, satisfy certain monotone properties with respect to $y_1, \cdots, y_n$. This paper is devoted to establishing the existence of other types of critical solutions and to a consideration of some of their properties.

2. **Notation.** Hereafter whenever the subscript i is used it is to be understood that i ranges over the set $(1, \cdots, n)$. Similarly, the sub-

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[1] A solution of (S), as used throughout this paper, is by definition a set of differentiable functions $y_1(x), \cdots, y_n(x)$ for which $y_i'(x) = f_i[x, y_1(x), \cdots, y_n(x)]$, $i=1, \cdots, n$, identically over some interval $a \leqq x \leqq b$.

[2] Here it is meant that each point of the curve in $(n+1)$-dimensional space given parametrically by $x=x$, $y_1=y_1(x), \cdots, y_n=y_n(x)$ lies in R over $x_0-h \leqq x \leqq x_0+k$. For the sake of brevity this terminology will be used freely.

[3] Osgood, *Beweis der Existenz einer Lösung der Differentialgleichung $dy/dx=f(x, y)$ ohne Hinzunahme der Cauchy-Lipschitz'schen Bedingung*, Monatshefte für Mathematik und Physik vol. 9 (1898) pp. 331–345.

[4] Montel, *Sur les suites infinies de fonctions*, Ann. École Norm. vol. 24 (1907) pp. 233–234.

[5] Perron, *Ein neuer Existenzbeweis für die Integrale der Differentialgleichung $y'=f(x, y)$*, Math. Ann. vol. 76 (1915) pp. 471–484.

[6] Kamke, *Zur Theorie der Systems gewöhnlicher Differentialgleichungen*, Acta Math. vol. 58 (1932) pp. 57–85.

scripts p and q will have the ranges $(1, \cdots, k)$ and $(k+1, \cdots, n)$, respectively.

For a fixed index k, $1 \leqq k \leqq n$, a set of functions $f_i(x, y_1, \cdots, y_n)$ will be said to have property (π_k) if, on the subset of R for all of whose points $x \geqq x_0$, the following conditions are satisfied:

(π_k)
(1) f_i is continuous in $(x, y_1, \cdots, y_n)$.
(2) f_p is monotone increasing in y_j, $j=1, \cdots, k$; $j \neq p$, and monotone decreasing in y_q.
(3) f_q is monotone decreasing in y_p and monotone increasing in y_j, $j=k+1, \cdots, n$; $j \neq q$.

In a similar manner (π_k^*) will designate that set of conditions one obtains on the subset of R for which $x \leqq x_0$ by interchanging the words "increasing" and "decreasing" in property (π_k).

To shorten further the notation $h_j[x, y]$ will be used to represent $h_j(x, y_1, \cdots, y_n)$, and $W(x)$ will be used to represent the set of n functions $w_1(x), w_2(x), \cdots, w_n(x)$. As in §1, P will designate a point $(x_0, y_{10}, \cdots, y_{n0})$ of a region R.

3. Preliminary theorems.

THEOREM 1. *Let the following conditions hold over the subset of a region R for all of whose points $x \geqq x_0$:*

(1) *The functions $f_i[x, y]$ are continuous in* $(x, y_1, \cdots, y_n)$.
(2) *The set of functions $g_i[x, y]$ has property* (π_k).
(3) $f_p[x, y] < g_p[x, y], f_q[x, y] > g_q[x, y]$.

Let $Y(x)$ be a solution of (S) *through P and $Z(x)$ be a solution of the system $y_i' = g_i[x, y]$ through P. If both of these solutions exist in R on $x_0 \leqq x < x_0 + a$, $a > 0$, then on $x_0 < x < x_0 + a$*

$$y_p(x) < z_p(x) \quad \text{and} \quad y_q(x) > z_q(x). \tag{H}$$

PROOF. Define $Q_p(x) = z_p(x) - y_p(x)$ and $Q_q(x) = y_q(x) - z_q(x)$. Then $Q_i(x_0) = 0$ and by applying hypothesis (3) one sees that

$$Q_p'(x_0) = g_p[x_0, z(x_0)] - f_p[x_0, y(x_0)] > 0$$

and

$$Q_q'(x_0) = f_q[x_0, y(x_0)] - g_q[x_0, z(x_0)] > 0.$$

Since $Q_i(x)$ is continuous on the right at x_0, it follows that (H) is true over some interval $x_0 < x < x_0 + \delta$, $\delta > 0$. Let γ be the upper limit of the set of values $x_0 + \delta$ for which (H) is true on $x_0 < x < x_0 + \delta$. Assume that γ is less than $x_0 + a$. Then since $Q_i(x)$ is continuous and has a derivative on $x_0 \leqq x < x_0 + a$, we have for at least one μ

(i) $$Q_\mu(\gamma) = 0,$$

(ii) $$Q_\mu'(\gamma) \leqq 0,$$

(iii) $$Q_i(\gamma) \geqq 0, \qquad i \neq \mu.$$

Relations (i) and (iii) give $z_p(\gamma) \geqq y_p(\gamma)$ and $z_q(\gamma) \leqq y_q(\gamma)$. By applying hypotheses (2) and (3) one obtains

(a) if $1 \leqq \mu \leqq k$, $f_\mu[\gamma, y(\gamma)] < g_\mu[\gamma, y(\gamma)] \leqq g_\mu[\gamma, z(\gamma)]$;

(b) if $k+1 \leqq \mu \leqq n$, $f_\mu[\gamma, y(\gamma)] > g_\mu[\gamma, y(\gamma)] \geqq g_\mu[\gamma, z(\gamma)]$.

From (ii) one obtains

(c) if $1 \leqq \mu \leqq k$, $z_\mu'(\gamma) \leqq y_\mu'(\gamma)$; i.e., $g_\mu[\gamma, z(\gamma)] \leqq f_\mu[\gamma, y(\gamma)]$;

(d) if $k+1 \leqq \mu \leqq n$, $y_\mu'(\gamma) \leqq z_\mu'(\gamma)$; i.e., $f_\mu[\gamma, y(\gamma)] \leqq g_\mu[\gamma, z(\gamma)]$.

Either (a) and (c) or (b) and (d) are contradictory. Hence $\gamma \geqq x_0 + a$.

COROLLARY. *The conclusion of Theorem* 1 *still follows if the conditions* (1) *and* (2) *are replaced by*:
(1′) *The functions* $g_i[x, y]$ *are continuous*;
(2′) *The set of functions* $f_i[x, y]$ *has property* (π_k).

PROOF. Proceeding as before one obtains

if $1 \leqq \mu \leqq k$, $f_\mu[\gamma, y(\gamma)] \leqq f_\mu[\gamma, z(\gamma)] < g_\mu[\gamma, z(\gamma)]$,

if $k+1 \leqq \mu \leqq n$, $f_\mu[\gamma, y(\gamma)] \geqq f_\mu[\gamma, z(\gamma)] > g_\mu[\gamma, z(\gamma)]$.

Thus the relations (a) and (b) still hold. The remainder of the proof is identical to the proof of the theorem.

THEOREM 2. *Let the following conditions hold over the subset of R for all of whose points* $x \leqq x_0$:
(1) *The functions* $f_i[x, y]$ *are continuous in* $(x, y_1, \cdots, y_n)$.
(2) *The set of functions* $g_i[x, y]$ *has property* (π_k^*).
(3) $f_p[x, y] > g_p[x, y]$, $f_q[x, y] < g_q[x, y]$.
Let $Y(x)$ *be a solution of* (S) *through P and* $Z(x)$ *be a solution of the system* $y_i' = g_i[x, y]$ *through P. If both of these solutions exist in R on* $x_0 - a < x \leqq x_0$, $a > 0$, *then on* $x_0 - a < x < x_0$

(H′) $$y_p(x) < z_p(x) \quad \text{and} \quad y_q(x) > z_q(x).$$

The proof of this theorem is analogous to the proof of Theorem 1. Obviously this theorem has a corollary similar to that of Theorem 1.

4. Definition of minimax solutions.

DEFINITION. *Let* $\Phi(x)$ *be a solution of* (S) *through P existing over an interval* I: $x_0 - h_1 < x < x_0 + h_2$ $(h_1, h_2 \geqq 0,\ h_1 + h_2 > 0)$ *such that for any* *other solution* $Y(x)$ *of* (S) *through* P *the relations*

$$\text{(A)} \qquad y_p(x) \leqq \phi_p(x), \qquad y_q(x) \geqq \phi_q(x)$$

or

$$\text{(B)} \qquad y_p(x) \geqq \phi_p(x), \qquad y_q(x) \leqq \phi_q(x)$$

hold over any subinterval of I *where* $Y(x)$ *exists. In case* (A) *the solution* $\Phi(x)$ *is designated as a* (k)max-$(n-k)$min solution *of* (S) *through* P *over* I *and in case* (B) *as a* (k) min-$(n-k)$ max solution *of* (S) *through* P *over* I. *In either case it is called a* minimax solution.

If $Y(x)$ is a (k)max-$(n-k)$min solution of (S) through P, it will be indicated as $U_1(x), \cdots, U_k(x), u_{k+1}(x), \cdots, u_n(x)$. Similarly $u_1(x), \cdots, u_k(x), U_{k+1}(x), \cdots, U_n(x)$ will designate a (k) min-$(n-k)$ max solution.

5. **Existence of minimax solutions.** Consider the systems

$$(\mathrm{S}_\epsilon) \qquad \begin{aligned} y_p' &= f_p(x, y_1, \cdots, y_n) + \epsilon, \\ y_q' &= f_q(x, y_1, \cdots, y_n) - \epsilon \end{aligned} \qquad (\epsilon > 0).$$

Hereafter the symbol $Y(x, \epsilon)$ will be used to designate the set of n functions $y_1(x, \epsilon), \cdots, y_n(x, \epsilon)$.

LEMMA 1. *Let the functions* $f_i[x, y]$ *be continuous in* R. *Then there exists an* $\epsilon_0 > 0$ *and an* $r > 0$ *such that all solutions of* (S) *and* (S_ϵ), *for* $0 < \epsilon < \epsilon_0$, *through* P, *exist in* R *on the interval* $|x - x_0| \leqq r$.

PROOF. Take any hyper-rectangle B: $|x-x_0| \leqq a$, $|y_i - y_{i0}| \leqq b$ lying in R. Let $Y(x)$ and $Y(x, \epsilon)$ be arbitrary solutions of (S) and (S_ϵ), respectively, through P. These solutions exist[7] to the boundary of R. Let M be the maximum value in B of $|f_i[x, y]|$. Take $\epsilon_0 = M$ and $r = \min(a, b/2M)$. For any x such that $|x - x_0| \leqq r$ and any ϵ such that $0 < \epsilon < \epsilon_0$

$$\begin{aligned} |y_i(x, \epsilon) - y_{i0}| &= \left| \int_{x_0}^{x} \{f_i[t, y(t, \epsilon)] \pm \epsilon\} dt \right| \\ &\leqq M|x - x_0| + \epsilon_0 |x - x_0| \leqq 2Mr \leqq b \end{aligned}$$

and

$$|y_i(x) - y_{i0}| = \left| \int_{x_0}^{x} f_i[t, y(t)] dt \right| \leqq Mr \leqq \frac{b}{2} \cdot$$

[7] Cf., for example, Kamke, *Differentialgleichungen reeller Funktionen*, Leipzig, 1930, p. 135.

THEOREM 3. *Let the set of functions* $f_i[x, y]$ *have property* (π_k). *Then the system* (S) *has a unique* (k)max-$(n-k)$min solution $U_1(x), \cdots, U_k(x), u_{k+1}(x), \cdots, u_n(x)$, *and a unique* (k)min-$(n-k)$max solution $u_1(x), \cdots, u_k(x), U_{k+1}(x), \cdots, U_n(x)$ *through* P. *These solutions exist at least on an interval* $x_0 \leqq x < x_0 + r$, $r > 0$.

PROOF. Use the notation of Lemma 1 so that on the interval I_r: $x_0 - r \leqq x \leqq x_0 + r$ any solution $Y(x)$ of (S) and $Y(x, \epsilon)$ of (S_ϵ) $(0 < \epsilon < \epsilon_0 = M)$ through P exists in R. Let j be any fixed index such that $1 \leqq j \leqq n$. The set of functions $\{y_j(x, \epsilon)\}$ is uniformly bounded and equicontinuous on I_r. For

$$y_j(x, \epsilon) = y_{j0} + \int_{x_0}^{x} f_j[t, y(t, \epsilon)]dt \pm (x - x_0)\epsilon,$$

$$| y_j(x, \epsilon) | \leqq | y_{j0} | + (M + \epsilon_0)r = | y_{j0} | + 2Mr$$

and

$$\begin{aligned} | y_j(x_2, \epsilon) - y_j(x_1, \epsilon) | &= \left| \int_{x_0}^{x_2} f_j[t, y(t, \epsilon)]dt \pm (x_2 - x_0)\epsilon \right. \\ &\quad \left. - \int_{x_0}^{x_1} f_j[t, y(t, \epsilon)]dt \mp (x_1 - x_0)\epsilon \right| \\ &= \left| \int_{x_1}^{x_2} f_j[t, y(t, \epsilon)]dt \pm (x_2 - x_1)\epsilon \right| \\ &\leqq 2M | x_2 - x_1 |. \end{aligned}$$

If ϵ_1 and ϵ_2 are chosen so that $0 < \epsilon_2 < \epsilon_1 < \epsilon_0$, then

$$f_p[x, y] + \epsilon_1 > f_p[x, y] + \epsilon_2, \quad \text{and} \quad f_q[x, y] - \epsilon_1 < f_q[x, y] - \epsilon_2.$$

A straightforward application of Theorem 1 gives

$$y_p(x, \epsilon_1) \geqq y_p(x, \epsilon_2) \quad \text{and} \quad y_q(x, \epsilon_1) \leqq y_q(x, \epsilon_2)$$

over $x_0 \leqq x \leqq x_0 + r$. Hence over this interval the functions $y_j(x, \epsilon)$ are monotone increasing functions of ϵ if $j = 1, \cdots, k$, and monotone decreasing functions of ϵ if $j = k+1, \cdots, n$. Choose a sequence $\{\epsilon_m\}$ such that $\epsilon_0 > \epsilon_1 > \epsilon_2 \cdots$ and $\lim_{m \to \infty} \epsilon_m = 0$. By the use of Ascoli's Theorem together with the monotone properties of the functions $y_j(x, \epsilon_m)$ one establishes the existence of a limiting function $y_j^*(x)$ to which the sequence $\{y_j(x, \epsilon_m)\}$ converges uniformly over $x_0 \leqq x \leqq x_0 + r$. By applying Theorem 1 again it is clear that $\lim_{\epsilon \to 0} y_i(x, \epsilon) = y_i^*(x)$ and the convergence is uniform over $x_0 \leqq x \leqq x_0 + r$.

The set of functions $y_1^*(x), \cdots, y_n^*(x)$ is a solution of (S) on

$x_0 \leqq x \leqq x_0+r$. For

$$y_i^*(x) = \lim_{\epsilon\to 0} y_i(x,\epsilon) = \lim_{\epsilon\to 0}\left\{y_{i0} + \int_{x_0}^{x} f_i[t, y(t,\epsilon)]dt \pm (x - x_0)\epsilon\right\}$$

$$= y_{i0} + \int_{x_0}^{x} \lim_{\epsilon\to 0} f_i[t, y(t,\epsilon)]dt$$

since $f_i[t, y(t, \epsilon)]$ converges uniformly to $f_i[t, y^*(t)]$ on $x_0 \leqq x \leqq x_0+r$. Then $y_i^*(x) = y_{i0} + \int_{x_0}^{x} f_i[t, y^*(t)]dt$, by the continuity of the functions $f_i[x, y]$.

Now consider an arbitrary solution $Y(x)$ of (S) through P. Notice that

$$f_p[x, y] < f_p[x, y] + \epsilon \quad \text{and} \quad f_q[x, y] > f_q[x, y] - \epsilon$$

and apply Theorem 1. One finds that over $x_0 \leqq x \leqq x_0+r$

$$y_p(x) \leqq y_p(x, \epsilon) \quad \text{and} \quad y_q(x) \geqq y_q(x, \epsilon).$$

Consequently

$$v_p(x) \leqq y_p^*(x) = \lim_{\epsilon\to 0} y_p(x, \epsilon) \quad \text{and} \quad y_q(x) \geqq y_q^*(x) = \lim_{\epsilon\to 0} y_q(x, \epsilon).$$

This proves that on $x_0 \leqq x < x_0+r$

$$y_p^*(x) = U_p(x) \quad \text{and} \quad y_q^*(x) = u_q(x).$$

The uniqueness of the solution $U_1(x), \cdots, U_k(x), u_{k+1}(x), \cdots, u_n(x)$ follows immediately from its definition.

The proof of the existence of a unique solution $u_1(x), \cdots, u_k(x), U_{k+1}(x), \cdots, U_n(x)$ is made in a similar manner by considering, instead of the system (S_ϵ), the system

$$(S_\epsilon') \qquad y_p' = f_p[x, y] - \epsilon, \qquad y_q' = f_q[x, y] + \epsilon.$$

THEOREM 4. *Let the set of functions $f_i[x, y]$ have property (π_k^*). Then the system* (S) *has a unique* (k)max-$(n-k)$min *solution,* $U_1(x), \cdots, U_k(x), u_{k+1}(x), \cdots, u_n(x)$, *and a unique* (k)min-$(n-k)$max solution $u_1(x), \cdots, u_k(x), U_{k+1}(x), \cdots, U_n(x)$, *through* P. *These solutions exist at least over an interval* $x_0 - r < x \leqq x_0$ $(r>0)$.

The proof of this theorem is analogous to that of Theorem 3.

EXAMPLE. Consider the system $y_1' = -5xy_2$, $y_2' = -3x(y_1)^{1/5}$. Let R be any region containing the point P: (0, 0). It is immediately seen that the functions $f_1(x, y_1, y_2) \equiv -5xy_2$ and $f_2(x, y_1, y_2) \equiv -3x(y_1)^{1/5}$ have properties (π_1) and (π_1^*) in R. For $x \geqq 0$ the given system has the solutions $U_1(x) = x^5$, $u_2(x) = -x^3$ and $u_1(x) = 0$, $U_2(x) = 0$. For $x \leqq 0$

it has the solutions $U_1(x) = -x^3$, $u_2(x) = x^5$ and $u_1(x) = 0$, $U_2(x) = 0$.

6. **Extension of minimax solutions to the boundary** R. The notation C_G will be used to represent the curve in $(n+1)$-dimensional space determined by the equations $x = x$, $y_i = g_i(x)$. Similarly $C_{G\epsilon}$ will be used to represent the curve determined by equations of the type $x = x$, $y_i = g_i(x, \epsilon)$. A curve of either type will be said to lie in R over an interval if for each value of x in this interval the corresponding point of the curve is in R.

We know from the proof of Theorem 3 that there exists some interval $x_0 \leqq x \leqq x_0 + r$ on which the functions $y_j(x, \epsilon)$ converge uniformly to a function $y_j^*(x)$ such that C_{Y^*} lies in R.

LEMMA 2. *Let the set of functions $f_i[x, y]$ have property (π_k). Let $Y(x, \epsilon)$ be a solution of* (S_ϵ) *through P, and X: (x_0, b), $b > x_0$, be the greatest interval to the right of x_0 over which, for all fixed indices j, $1 \leqq j \leqq n$, the functions $y_j(x, \epsilon)$ converge uniformly (as ϵ approaches zero) to a function $y_j^*(x)$ so that C_{Y^*} lies in R. Then the point $[b, y_1^*(b), \cdots, y_n^*(b)]$ lies in the boundary of R.*

PROOF. It will first be proved that X cannot be a closed interval. If so, C_{Y^*} exists in R over $x_0 \leqq x \leqq b$ and $[b, y_1^*(b), \cdots, y_n^*(b)]$ lies in R. Choose a hyper-rectangle B: $|x - b| \leqq \alpha$, $|y_i - y_i^*(b)| \leqq \beta$ $(0 < \alpha < b - x_0, \beta > 0)$ lying in R. By the property of uniform convergence there exists an $\epsilon_1 > 0$ such that for $\epsilon < \epsilon_1$ the curves C_{Y_ϵ} lie in R and also

$$| y_i(x, \epsilon) - y_i^*(x) | < \beta/2 \quad \text{on} \quad x_0 \leqq x \leqq b.$$

The curves C_{Y_ϵ} are known to exist to the boundary of R. Now let $M = \max \{ |f_i(x, y_1, \cdots, y_n)| + |\epsilon_1| \}$ in B. Then if $\epsilon < \epsilon_1$, C_{Y_ϵ} exists in B over $b \leqq x \leqq b + \beta/2M$. For over this interval

$$\begin{aligned} | y_i(x, \epsilon) - y_i^*(b) | &\leqq | y_i(x, \epsilon) - y_i(b, \epsilon) | + | y_i(b, \epsilon) - y_i^*(b) | \\ &\leqq M | x - b | + \beta/2 \leqq \beta. \end{aligned}$$

Thus the curves C_{Y_ϵ}, $\epsilon < \epsilon_1$, exist in R over the interval $x_0 \leqq x \leqq b + \beta/2M$. It can be shown as in the proof of Theorem 3 that over this interval, for a fixed j, the set $\{y_j(x, \epsilon)\}$ is uniformly bounded and equicontinuous and monotone increasing in ϵ if $j = 1, \cdots, k$, monotone decreasing in ϵ if $j = k+1, \cdots, n$. Consequently $\lim_{\epsilon \to 0} y_j(x, \epsilon)$ exists uniformly over $(x_0, x_0 + b + \beta/2M)$. This contradicts the definition of X.

Now assume that X is the interval $x_0 \leqq x < b$; i.e., X is open on the right. If $[b, y_1^*(b), \cdots, y_n^*(b)]$ lies in R, where by definition $y_i^*(b) = \lim_{x \to b^-} y_i^*(x)$, then $\lim_{\epsilon \to 0} y_i(x, \epsilon) = y_i^*(x)$ uniformly over the closed interval $x_0 \leqq x \leqq b$. For, if ϵ is sufficiently small, we have for an arbitrary $\eta > 0$

$$|y_i(x, \epsilon) - y_i^*(x)| < \eta/2 < \eta \quad \text{on} \quad x_0 \leqq x < b.$$

Consequently,

$$|y_i(b, \epsilon) - y_i^*(b)| \leqq \eta/2 < \eta$$

and

$$|y_i(x, \epsilon) - y_i^*(x)| < \eta \quad \text{on} \quad x_0 \leqq x \leqq b.$$

This again contradicts the fact that X is by definition the greatest interval such that C_{Y^*} lies in R. Hence the point $[b, y_1^*(b), \cdots, y_n^*(b)\}$ lies in the boundary of R.

An argument entirely similar to that used in the proof of Theorem 3 proves that the set of functions $y_1^*(x), \cdots, y_n^*(x)$ constitutes a solution of (S) over $x_0 \leqq x < b$. Now let $Y(x)$ be an arbitrary solution of (S) through P existing in R on $x_0 \leqq x < \bar{b} < b$. If ϵ is taken sufficiently small to insure the existence in R of C_{Y_ϵ} on $x_0 \leqq x \leqq \bar{b}$, an application of Theorem 1 shows that over this interval

$$y_p(x) \leqq y_p(x, \epsilon) \quad \text{and} \quad y_q(x) \geqq y_q(x, \epsilon),$$

and hence $y_p(x) \leqq y_p^*(x)$ and $y_q(x) \geqq y_q^*(x)$. This argument together with an obvious modification based on the system

$$y_p' = f_p[x, y] - \epsilon, \qquad y_q' = f_q[x, y] + \epsilon$$

proves the following:

THEOREM 5. *Let the set of functions $f_i[x, y]$ have property (π_k). Then the solutions $U_1(x), \cdots, U_k(x), u_{k+1}(x), \cdots, u_n(x)$ and $u_1(x), \cdots, u_k(x), U_{k+1}(x), \cdots, U_n(x)$ of* (S) *through P exist for $x \geqq x_0$ to the boundary of R.*

Analogous to Theorem 5 we have:

THEOREM 6. *Let the set of functions $f_i[x, y]$ have property (π_k^*). Then the solutions $U_1(x), \cdots, U_k(x), u_{k+1}(x), \cdots, u_n(x)$ and $u_1(x), \cdots, u_k(x), U_{k+1}(x), \cdots, U_n(x)$ of* (S) *through P exist for $x \leqq x_0$ to the boundary of R.*

7. Approximation of minimax solutions by sets of under-over functions.

DEFINITION.[8] *With reference to* (S) *and P a set of (k)over-*

[8] The sets of under-over functions defined here are generalizations of the under and over functions originated by O. Perron, loc. cit.

$(n-k)$under functions is a set of n functions $\phi_1(x), \cdots, \phi_n(x)$ which are finite and continuous over an interval $x_0 \leqq x \leqq x_0+a$, $a>0$, which have left- and right-hand derivatives over this interval, such that

$$\phi_i(x_0) = y_{i0}, \quad D_\pm \phi_p(x) > f_p[x, \phi(x)] \quad \text{and} \quad D_\pm \phi_q(x) < f_q[x, \phi(x)].$$

Similarly $\phi_1(x), \cdots, \phi_n(x)$ constitutes a set of (k)under-$(n-k)$over functions if the above conditions hold except that

$$D_\pm \phi_p(x) < f_p[x, \phi(x)] \quad \text{and} \quad D_\pm \phi_q(x) > f_q[x, \phi(x)].$$

THEOREM 7. *Let the functions $f_i[x, y]$ have property (π_k). Let I: $x_0 \leqq x \leqq x_0+a$, $a>0$, be an interval over which the minimax solutions of* (S) *through P determine curves which lie in R. Then the solution $U_1(x), \cdots, U_k(x), u_{k+1}(x), \cdots, u_n(x)$ can be uniformly approximated over I by a set of (k)over-$(n-k)$under functions. Similarly the solution $u_1(x), \cdots, u_k(x), U_{k+1}(x), \cdots, U_n(x)$ can be uniformly approximated over I by a set of (k)under-$(n-k)$over functions.*[9]

PROOF. A solution of (S_ϵ) through P constitutes a set of (k)over-$(n-k)$under functions. A solution of (S'_ϵ) through P constitutes a set of (k)under-$(n-k)$over functions. It has already been shown (in the proofs of Theorem 3 and Lemma 2) that over I solutions of (S_ϵ) and (S'_ϵ) uniformly approach the minimax solutions as limiting functions.

8. Other properties of minimax solutions.

DEFINITION. *Assume that all solutions of* (S) *through P exist in R over I: $x_0-h<x<x_0+k$ $(h \geqq 0,\ k \geqq 0)$. The* tube of solutions; *T_P of* (S) *through P over I is the set of all points $[x, y_1(x), \cdots, y_n(x)]$ in $(n+1)$-dimensional space belonging to any solution through P and having an abscissa x such that $x_0-h<x<x_0+k$.*

Let T_P exist in R on an interval I: $a<x<b$. Due to the strict inequality of relations (H) and (H′) of Theorems 1 and 2, respectively, we know that for an arbitrary solution $Y(x)$ of (S) and $Y(x, \epsilon)$ of (S_ϵ)

$$y_i(x) < y_i(x, \epsilon) \quad \text{or} \quad y_i(x) > y_i(x, \epsilon)$$

for x in I, $x \neq x_0$, $\epsilon>0$. Thus no point of any curve C_{Y_ϵ}, having an abscissa $x \neq x_0$, is in T_P. But for ϵ sufficiently small, points of such curves are arbitrarily near any point of a curve lying in T_P and determined by a minimax solution. This proves:

THEOREM 8. *Under the hypothesis (π_k) and/or the hypothesis (π_k^*)*

[9] The definitions of sets of under-over functions can be modified so that a similar theorem can be stated with respect to an interval $x_0-a \leqq x \leqq x_0$, $a>0$.

the curve in R determined by a minimax solution of (S) *through P lies entirely in the boundary of T_P.*

Now let $T_P(b)$ be the intersection of T_P with the hyperplane $x=b$. $T_P(b)$ is a continuum by a theorem of H. Kneser.[10]

THEOREM 9. *If j is any fixed index, $1 \leqq j \leqq n$, and B_j is a number such that*

$$\text{g.l.b. } \{y_j(b)\} < B_j < \text{l.u.b. } \{y_j(b)\},$$

where $\{y_j(b)\}$ is the set of all $y_j(b)$ values in $T_P(b)$, then there exists a solution of (S) *such that $y_j(b)=B_j$.*[11]

PROOF. Otherwise the intersection of the hyperplanes $x=b$, $y_j=b_j$ divides the continuum $T_P(b)$ into two disjoint, nonempty sets which have no limit point in common.

COROLLARY 1. *If the set of functions $f_i[x, y]$ satisfies the hypothesis (π_k) or (π_k^*), B_j is a number such that $u_j(b) \leqq B_j \leqq U_j(b)$.*

COROLLARY 2. *There exists a solution of* (S) *satisfying the $n+1$ boundary conditions $y_i(x_0)=y_{i0}$, $y_j(b)=B_j$.*

Finally, one may observe that the results of this paper, together with those of the papers cited, exhibit a strong similarity between critical point theory for functions of several real variables and critical solution theory for systems of ordinary differential equations. It is believed that further comparison of the two theories will yield other striking analogies.

THE UNIVERSITY OF CALIFORNIA, DAVIS AND
THE UNIVERSITY OF NORTH CAROLINA

[10] Preuss. Akad. Wiss. Sitzungsber. Phys.-Math. Klasse, 1923. Another proof has been given by M. Müller, Math. Zeit. vol. 28 (1928) pp. 349–355.

[11] This is a generalization of a theorem of G. Mie which can be stated as follows: Let $f(x, y)$ be continuous over a region R of the xy-plane and P: (x_0, y_0) be a point of R. Let $u(x)$ be the minimum solution through P and $U(x)$ be the maximum solution through P of the equation $y'=f(x, y)$. Let $u(x)$ and $U(x)$ exist in R over the interval $x_0-h<x<x_0+k$, where $h\geqq 0$, $k\geqq 0$. Then if Q: (ξ, η) is a point such that $x_0-h<\xi<x_0+k$, $u(\eta)\leqq \eta \leqq U(\eta)$, there exists at least one solution of $y'=f(x, y)$ through P and Q.