PRODUCTS OF ORDERED SYSTEMS

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1. **Introduction.** Some miscellaneous results concerning ordered products of ordered systems are given in this paper. For example, it is shown that the only products that are complemented or relatively complemented lattices are cardinal products. Conditions for a product of lattices to be a lattice and for a product of partially ordered sets to satisfy the ascending chain condition are given.

By making obvious substitutions, one can obtain a number of corollaries to the theorems stated giving facts relating to the ordinal product and cardinal product of two ordered systems and the ordinal power of an ordered system. Also results concerning ordered sums analogous to many of the results given here could be worked out easily.

Precisely the same notation and terminology used in [2][1] will be used in this paper. However, for the sake of convenience we list here some of the definitions and symbols that will be employed. By an *ordered system* is meant a nonempty set R of elements in which a reflexive binary relation $r \geqq r'$ is defined. Unless otherwise specified, an italic capital letter always will denote an ordered system in the sequel. A *subsystem* T of R is a subset of elements of R with the order relation in T imposed by that in R.

The expressions and symbols *maximal element*, *ascending chain condition*, $>$, etc., will have their usual meanings (see [1], for example). The symbols $\vee$ and $\wedge$ will be used in denoting least upper bound (l.u.b.) and greatest lower bound (g.l.b.) respectively. The symbols 0 and I will denote bounds of bounded ordered systems. The term *number* will mean partially ordered set.

If for each element r in R, S_r is an ordered system, the *ordered product* over R of the S_r (denoted by $\prod_R S_r$) is the system P where the elements of P are the functions f defined on R such that $f(r) \in S_r$, while $f \geqq f'$ means that if $f(r) \neq f'(r)$, there exists $r' \geqq r$ such that $f(r') > f'(r')$.

In order that the statements of theorems may be simplified, throughout the sequel it will be assumed that every factor, S_r, in a product, $\prod_R S_r$, contains more than one element.

Several theorems in [3], which also are stated in the introduction in [2], will be used in proofs in this paper.

Presented to the Society, April 25, 1952; received by the editors February 20, 1952.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

2. **Products that satisfy chain conditions.** By making use of all of the conditions of [3, 4.13] we could state a theorem slightly more general than the one below. However, in order to simplify the theorem, we shall make the assumption that R is a number.

THEOREM 1. *Let R be a number. Then $\prod_R S_r$ is a number satisfying the ascending (descending) chain condition if and only if all S_r are numbers satisfying the ascending (descending) chain condition, and the subsystem of R of all r such that S_r is not a cardinal is finite.*

PROOF. Throughout this proof we let R_1 denote the subsystem of R of all r such that S_r is not a cardinal. R_1 may be an empty set.

Assume that $\prod_R S_r$ is a number satisfying the ascending (descending) chain condition. Then [3, 3.9] implies that all S_r are numbers satisfying the ascending (descending) chain condition.

Suppose that R_1 is not finite. Let $r_1, r_2, \cdots, r_i, \cdots, 1 \leqq i < \infty$, be distinct elements of R_1. Let $s_i > t_i$ in S_{r_i} for $1 \leqq i < \infty$. We now describe elements f_j, $1 \leqq j < \infty$, in $\prod_R S_r$ as follows. Let $f_j(r) = f_k(r)$ for r in R, $r \neq r_i$, $1 \leqq i < \infty$, for $1 \leqq j$, $k < \infty$. For each i, $1 \leqq i < \infty$, let $f_j(r_i) = t_i(s_i)$ for $1 \leqq j \leqq i$, and let $f_j(r_i) = s_i(t_i)$ for $i < j < \infty$. Then $f_1 < f_2 < \cdots < f_j < \cdots$ $(f_1 > f_2 > \cdots > f_j > \cdots)$ for $1 \leqq j < \infty$, a contradiction to our hypothesis. Hence R_1 is finite.

Now assume that all S_r are numbers satisfying the ascending (descending) chain condition and that R_1 is finite. Then [3, 4.13] implies that $\prod_R S_r$ is a number.

If R_1 is empty, $\prod_R S_r$ contains no two distinct comparable elements. Assume that R_1 is not empty. Suppose that there is a chain, $f_1 < f_2 < \cdots < f_i < \cdots$ $(f_1 > f_2 > \cdots > f_i > \cdots)$, $1 \leqq i < \infty$, in $\prod_R S_r$. Let r_i for each i, $1 \leqq i < \infty$, be a fixed element in R_1 such that $f_i(r_i) < f_j(r_i)$ $(f_i(r_i) > f_j(r_i))$ for some $j > i$ but such that for any element $r > r_i$ in R_1, $f_i(r) = f_j(r)$ for $i \leqq j < \infty$. Each element r_i exists since R_1 is finite. Also for this reason, there is an element x in R_1 such that $r_i = x$ for an infinite number of values of i, say $n(1) < n(2) < \cdots < n(k) < \cdots$, $1 \leqq k < \infty$.

Since S_x satisfies the ascending (descending) chain condition, there is an element s in S_x such that $f_{n(k)}(x)$ equals s for at least one value of k, say q, and such that $f_{n(k)}(x)$ is not greater (less) than s for any value of k. $f_{n(q)}(x) = s < f_p(x)$ $(f_{n(q)}(x) > f_p(x))$ for some integer $p > n(q)$. Let t be an integer such that $n(t) > p$. For any $r > x$ in R_1, $f_{n(q)}(r) = f_p(r) = f_{n(t)}(r)$. But $f_p < f_{n(t)}$ $(f_p > f_{n(t)})$. Hence $s < f_p(x) \leqq f_{n(t)}(x)$ $(s > f_p(x) \geqq f_{n(t)}(x))$, a contradiction. And so $\prod_R S_r$ satisfies the ascending (descending) chain condition.

3. **Products that are lattices.** A product, $\prod_R S_r$, can be a lattice

without every factor S_r being a lattice (see [2, Corollary]). But if the assumption that every S_r is a lattice is not made, a set of necessary and sufficient conditions that $\prod_R S_r$ be a lattice is very inelegant.

THEOREM 2. *For every element r in R, let S_r be a lattice. Then $\prod_R S_r$ is a lattice if and only if the following conditions are satisfied.*

(1) *R is a number satisfying the ascending chain condition.*

(2) *For any element r in R, S_r is bounded if either there is a proper successor of r, say t, such that S_t is not a chain or there exist in R two incomparable successors of r.*

PROOF. Assume $\prod_R S_r$ is a lattice. Then by [3, 4.13], (1) holds.

Suppose that there is a proper successor t of some element r in R such that S_t is not a chain. Let a and b be two incomparable elements of S_t. Let $f_1(t)=a$, $f_2(t)=b$, and $f_1(w)=f_2(w)$ for w in R, $w\neq t$. Let $f=f_1\vee f_2$ in $\prod_R S_r$. Then $f(v)>f_i(v)$, $i=1, 2$, for some v in R greater than or equal to t. Now suppose that S_r does not contain 0. Let $k(w)=f(w)$ for w in R, $w\neq r$. Let $k(r)$ be an element that is not greater than or equal to $f(r)$. Then $k>f_i$, $i=1, 2$, but k is not greater than f, a contradiction. Hence $0\in S_r$. By a dual argument, $I\in S_r$.

Now suppose that there exist two incomparable successors, u and v, in R of an element r in R. Let $f_1(u)>f_2(u)$, $f_2(v)>f_1(v)$, and $f_1(w)=f_2(w)$ for w in R, $w\neq u$, $w\neq v$. Let $f=f_1\vee f_2$. Then $f(z)>f_2(z)$ for some $z\geqq u$ in R, and $f(y)>f_1(y)$ for some $y\geqq v$ in R. Hence again it can be shown that S_r must contain 0. By a dual proof, S_r contains I.

Now assume that (1) and (2) hold. Then [3, 4.13] implies that $\prod_R S_r$ is a number.

Let f_1 and f_2 be any two elements in $\prod_R S_r$. Let $g(r)=f_1(r)\vee f_2(r)$ for each element r in R. Let E_i, $i=1, 2$, be the subsystem of all r of R such that there exists $r'\geqq r$ for which $g(r')>f_i(r')$.

(Throughout the rest of this proof, if S is any ordered system, the symbol S' will denote the subsystem of S of all the maximal elements of S; if S and T are any two subsystems of any given ordered system, the symbol $S-T$ will denote the subsystem of all the elements of S that are not also in T. The symbols $\cup$ and $\cap$ will have the usual meanings of union and intersection respectively.)

It is easily verified that $(E_1\cup E_2)'=(E_1'\cap E_2')\cup(E_1'-E_2)\cup(E_2'-E_1)$.

For w in $(E_1\cup E_2)'$, let $f(w)=g(w)$. Now let w be in $E_1\cup E_2$ but not in $(E_1\cup E_2)'$. If there is a successor of w in $E_1'\cap E_2'$ or if w has a successor in $E_1'-E_2$ and one in $E_2'-E_1$, let $f(w)=0$. If $f(w)$ has not been defined yet and w has a successor in $E_1'-E_2$, let $f(w)=f_2(w)$. For any remaining w in $E_1\cup E_2$, let $f(w)=f_1(w)$.

For w in $(R-E_1)-E_2$, let $f(w)=f_1(w)=f_2(w)$.

By merely using the definition of the l.u.b. of two elements, one can check without difficulty that $f=f_1\vee f_2$. A dual proof shows that $\prod_R S_r$ is a lattice.

LEMMA. *If $\prod_R S_r$ is a lattice, no factor S_r is a cardinal.*

PROOF. Suppose that for some element r in R, S_r is a cardinal. Let a and b be two distinct elements in S_r. Let $f_1(r)=a$, $f_2(r)=b$, and $f_1(w)=f_2(w)$ for $w\neq r$ in R. Let $f=f_1\vee f_2$. Let $k(w)=f(w)$ for $w\neq r$, and let $k(r)$ be an element not equal to $f(r)$. Then $k\geqq f_1$ and $k\geqq f_2$, but k is not greater than f, a contradiction.

Definitions of the terms *complemented lattice*, *relatively complemented lattice*, *Boolean algebra*, etc., can be found in [1].

THEOREM 3. *$\prod_R S_r$ is a complemented (relatively complemented) lattice if and only if R is a cardinal and every S_r is a complemented (relatively complemented) lattice.*

PROOF. Assume that $\prod_R S_r$ is a complemented (relatively complemented) lattice. Then [3, 4.13] implies that all S_r are numbers. Also it follows from the lemma that no S_r is a cardinal, and so by [3, 4.13], R is a number satisfying the ascending chain condition.

Suppose that R is not a cardinal. Then there is an element x in R that is less than a maximal element, say r, of R. (If $\prod_R S_r$ is a complemented lattice, then by [2, p. 899, (4)], every factor is bounded.) Let $f_1(w)=I$ and $f_2(w)=0$ (let $f_1(w)>f_2(w)$) for each element w in R. Let $f(r)=f_1(r)$, and let $f(w)=f_2(w)$ for $w\neq r$ in R. Let g be a complement (relative complement) of f in the closed interval $[f_2, f_1]$. Since $f\vee g=f_1$ and $f\wedge g=f_2$, it follows that $f_1(r)\geqq g(r)\geqq f_2(r)$.

Suppose that $g(r)>f_2(r)$. Let $k(x)=f_1(x)$, and let $k(w)=f_2(w)$ for $w\neq x$ in R. Then $k<f$. By using the fact that $g>f_2$, one can verify easily that $k<g$. But $k>f_2$, a contradiction. Hence $g(r)=f_2(r)$.

Let $h(x)=f(x)=f_2(x)$, and let $h(w)=f_1(w)$ for w in R, $w\neq x$. Then $h\geqq f$. By using the fact that $f_1>g$, one can check readily that $h>g$. But $h<f_1$, a contradiction. Consequently R is a cardinal.

It is now not difficult to verify that every S_r is a complemented (relatively complemented) lattice.

If R is a cardinal and every S_r is a complemented (relatively complemented) lattice, it can be checked easily that $\prod_R S_r$ is a complemented (relatively complemented) lattice.

The following corollary can be verified without difficulty.

COROLLARY. *$\prod_R S_r$ is a Boolean algebra if and only if R is a cardinal and every S_r is a Boolean algebra.*

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